UBC Math Circle 2021 Problem Set 1

1. (a) Let p be a prime number. Show that $\Phi_p(x) = \frac{x^p-1}{x-1} \in \mathbb{Z}[x]$ is irreducible.

   (b) Show that $\Phi_{p^r}(x) = \frac{x^{p^r}-1}{x^{p^{r-1}}-1} \in \mathbb{Z}[x]$ is irreducible for any $r \in \mathbb{N}$.

2. Show for $n \geq 5$ that $1 + \prod_{i=1}^{n}(x-i) \in \mathbb{Z}[x]$ is irreducible.

3. Show that $x^4 + 1$ is irreducible in $\mathbb{Z}[x]$ but not in $\mathbb{Z}/p\mathbb{Z}[x]$ for any prime p.

4. (a) Show that for a prime $p \equiv 3 \pmod 4$, $x^2 + y^2 \equiv 0 \pmod p$ implies $x \equiv y \equiv 0 \pmod p$

   (b) Find the solutions to the following congruence:

$$2x^2 + 6xy - 2x + 5y^2 - 4y + 1 \equiv 0 \pmod{2021}$$

5. (QM-AM-GM-HM inequality) Let $x_1, \dots, x_n$ be n positive real numbers. Then

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot \cdots \cdot x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

6. Let a, b, c be positive real numbers. Show that

$$\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

7. Let x, y, z be positive reals satisfying $x^2 + y^2 + z^2 = 2xy + 2xz + 2yz$. Prove that

$$\frac{x+y+z}{3} \geq \sqrt[3]{2xyz}.$$

8. Let $n \geq 2$ be an integer. Prove that

$$n\left(1 - \frac{1}{\sqrt[n]{n}}\right) + 1 > 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > n(\sqrt[n]{n+1} - 1).$$

9. (Complementary Beatty sequences) Let α and β be positive irrational numbers such that $1/\alpha + 1/\beta = 1$. Show that the sequences $a_n = \lfloor \alpha n \rfloor$ and $b_n = \lfloor \beta n \rfloor$ are disjoint and their union is $\mathbb{N}$.