Undergraduate Mathematics Society (UMS)

Mathematics Student Union (MSU)

TRUSU Math Club

CLASH

OF

EQUATIONS

Undergraduate
Mathematics
Competition

July 8th
2023

Regulations

- There are 3 sections in this contest:
  1. Calculus
  2. Linear Algebra
  3. Discrete Mathematics
- You (and your team) must pick one question from each of three sections. In the event two problems from the same sections are submitted, the one to grade will be chosen at random.
- The contest is open book, and you are allowed to use the internet; however, you should not communicate with anyone outside of your registered team about the exam.
- The solution can include any material taught in a math undergraduate course, but cannot include material from pure graduate courses.
- Note that if you are submitting your solutions in handwriting, you are responsible for your writing being legible and readable.
- Solutions must be submitted before 3 : 30 PM, however, you will have a grace period of 5 minutes in case of unplanned internet issues. This being said, nothing is going to be accepted after 3 : 35 PM.
- The contest organizers will be able to answer logistical questions on Zoom during the contest time.
- Please do not include your names in your submission for fair grading. You should only include your team name.
- The grading done by the jury is final, and no regrading requests will be accepted.

Submission Instructions

You should submit your problems as a single pdf through a *Google Form* Submission Form. Note that you will be required to sign in to your *Google* account.

GOOD LUCK!

CALCULUS RELATED QUESTIONS

Problem 1

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the **intermediate value property** if for any $a < b$ and any u strictly between $f(a)$ and $f(b)$, there is a c with $a < c < b$ and $f(c) = u$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the intermediate value property such that for some $n \geq 1$, $f^{(n)}(x) = -x$ for all x (where $f^{(n)}$ denotes the n-fold composition $f \circ f \circ \ldots \circ f$).

Problem 2

Find the value of the integral below or prove that it diverges:

$$\int_0^{+\infty} \exp(-x^2 \sin x^2)\, dx.$$

LINEAR ALGEBRA QUESTIONS

by Saeed Rahmati

Problem 1

An n-dimensional vector space V and a linear transformation $\theta : V \to V$ are given. Consider the powers of θ, i.e. $1, \theta, \theta^2, \ldots$. Prove that there exists a nonzero integer s such that

$$V = im(\theta^s) \oplus ker(\theta^s).$$

Problem 2

Let $A = (a_{ij})$ be an $n \times n$ matrix over the field of real numbers such that for all i we have $\sum_{j=1}^{n} a_{ij} = a$. Consider a natural number $n = 2, 3, \ldots$. If $A^n = I$, find all possible values of a.

DISCRETE MATH QUESTIONS

by Jozsef Solymosi

Problem 1

Let $G_n = [n] \times [n]$ denote the points of the integer grid, i.e.

$$G_n = \{(a, b) | a, b \in \mathbb{N}, 1 \leq a, b \leq n\}.$$

Prove the following statement: If we have a point set $S \subset G_n$ and

$$|S| \geq (2n)^{\frac{3}{2}}$$

then S contains five points,

$$p_1 = (a_1, b_1), p_2 = (a_2, b_2), p_3 = (a_3, b_3), p_4 = (a_4, b_4), p_5 = (a_5, b_5)$$

such that $a_1 = a_2, b_1 = b_4, a_4 = a_5, b_2 = b_3$ and $a_3 + b_3 = a_5 + b_5$.

Problem 2

Let's write 100 as the sum of 50 numbers, where the summands are integers between one and 50.

$$x_1 + x_2 + \ldots + x_{49} + x_{50} = 100 \quad x_i \in \mathbb{N}, \quad 1 \leq x_i \leq 50.$$

An $I \subset [50] = \{1, 2, 3, \ldots, 49, 50\}$ is called a *halving partition set* if the x_i-s can be partitioned into two sets, both having the same sum, 50.

$$\sum_{i \in I} x_i = \sum_{i \in \{[50] \setminus I\}} x_i.$$

Prove that there is always a halving partition set.