Undergraduate Mathematics Society (UMS)

Mathematics Student Union (MSU)

TRUSU Math Club

CLASH
OF
EQUATIONS

SOLUTIONS

Only intended solutions,

which were of course not used by the participants...

Undergraduate
Mathematics
Competition

July 8th
2023

CALCULUS RELATED QUESTIONS

Problem 1

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the **intermediate value property** if for any $a < b$ and any u strictly between $f(a)$ and $f(b)$, there is a c with $a < c < b$ and $f(c) = u$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the intermediate value property such that for some $n \geq 1$, $f^{(n)}(x) = -x$ for all x (where $f^{(n)}$ denotes the n-fold composition $f \circ f \circ \ldots \circ f$).

SOLUTION

The only solution which satisfies the conditions of problem 1 is the function $f(x) = -x$. The proof will make use of the following Lemma.

Lemma 0.1. *Let $f : [a, b] \to \mathbb{R}$ be a one-to-one function which has the intermediate value property. Then f is strictly monotone.*

Proof. Without loss of generality, we may assume that $f(a) < f(b)$. Assume f is not strictly increasing. Then there are $c, d \in [a, b]$ such that $c < d$ but $f(c) \geq f(d)$. Since f is injective, $f(c) > f(d)$. Suppose for contradiction that $f(a) > f(c)$. Then $f(c) < f(a) < f(b)$ and there is an $a' \in [b, c]$ such that $f(a') = f(a)$, contradicting injectivity. We can similarly exclude $f(a) > f(d)$. Thus $f(a) \leq f(c)$ and $f(a) \leq f(d)$. Note that $a \neq d$ since this would imply that $f(c) = f(d) = f(a)$. Whence we conclude by injectivity that $f(c) > f(d) > f(a)$ and the intermediate value property shows that there is a $d' \in [a, c]$ such that $f(d') = f(d)$, contradicting injectivity. Hence f is strictly monotone. □

Let f satisfy the iterative property in the statement of the problem. The condition implies that $f^{(n)}$ (and therefore also f) is a bijection. Hence by the earlier lemma, f is strictly monotone. Since $-x$ is strictly decreasing and f is strictly monotone, we conclude that f is also strictly decreasing. Furthermore

$$f(-x) = f[f^{(n)}(x)] = f^{(n)}[f(x)] = -f(x)$$

so f is odd. In particular, $f(0) = 0$. Combining this with the fact that f is strictly decreasing implies that $xf(x) < 0$ for all $x \neq 0$. Indeed, if $x > 0$, then $f(x) < f(0) = 0$, and conversely if $x < 0$.

Now pick an $x_0 > 0$ and let $x_k = f(x_{k-1})$. Then $x_n = -x_0$. We claim that $(-1)^k x_k > 0$. Indeed, the base case $k = 0$ is clear, hence suppose the inequality has been proved for some $k \geq 0$. Then $x_k x_{k+1} < 0$. Since

$(-1)^k x_k > 0$ it is immediate that $(-1)^{k+1}x_{k+1} > 0$, completing the induction. In particular, $(-1)^n x_n = (-1)^{n+1}x_0 > 0$, so n is odd.

Finally, assume that $x_1 > -x_0$. Since f is decreasing and odd,

$$x_2 = f(x_1) < f(-x_0) = -f(x_0) = -x_1$$

so we obtain inductively that $(-1)^k x_k > (-1)^{k+1}x_{k+1}$. This yields $x_0 > -x_n$, a contradiction. Similarly, the assumption $x_1 < -x_0$ leads to the contradiction $x_n < -x_0$. Hence it follows that $x_1 = -x_0$ and $f(x) = x$ for all $x > 0$. Since f is odd, this implies immediately that $f(x) = -x$ for all x.

Problem 2

Find the value of the integral below or prove that it diverges:

$$\int_0^{+\infty} \exp(-x^2 \sin x^2)\, dx.$$

SOLUTION

Let's prove that the integral diverges. By definition of an improper integral, one has

$$\int_0^{+\infty} \exp(-x^2 \sin x^2)\, dx = \lim_{A\to\infty} \int_0^{A} \exp(-x^2 \sin x^2)\, dx. \tag{1}$$

Let N be the largest integer such that $\pi N \le A$. Since the function under the integral is positive, we also have

$$\int_0^{A} \exp(-x^2 \sin x^2)\, dx \ge \int_0^{\pi N} \exp(-x^2 \sin x^2)\, dx. \tag{2}$$

For $N = 1, 2, \ldots$ one also has

$$\int_0^{\pi N} \exp(-x^2 \sin x^2)\, dx = \sum_{n=1}^{N} \int_{\pi(n-1)}^{\pi n} \exp(-x^2 \sin x^2)\, dx. \tag{3}$$

Consider an integral $\int_{\pi(n-1)}^{\pi n} \exp(-x^2 \sin x^2)\, dx$ instead. Substitute $x = \pi n - t$, then

$$\int_{\pi(n-1)}^{\pi n} \exp(-x^2 \sin x^2)\, dx = \int_0^{\pi} \exp(-(\pi n - t)^2 \sin t^2)\, dt. \tag{4}$$

Recalling the inequality $|\sin t| \leq |t|$, which is true for all real numbers t, one gets $(\pi n - t)^2 \sin t^2 \leq (\pi n - t)^2 t^2 \leq (\pi n)^2 t^2$, which leads to

$$\int_0^{\pi} \exp(-(\pi n - t)^2 \sin t^2)\, dt \geq \int_0^{\pi} \exp(-\pi^2 n^2 t^2)\, dt. \tag{5}$$

Let's substitute $\pi n t = \xi$, then we get

$$\int_0^{\pi} \exp(-\pi^2 n^2 t^2)\, d = \frac{1}{\pi n} \int_0^{\pi^2 n} \exp(-\xi^2)\, d\xi \geq \frac{1}{\pi n} \int_0^{\pi^2} \exp(-\xi^2)\, d\xi. \tag{6}$$

Let $C = \int_0^{\pi^2} \exp(-\xi^2)\, d\xi$, which is some positive real number. from the inequalities above, one gets

$$\int_0^{A} \exp(-x^2 \sin x^2)\, dx \geq \sum_{n=1}^{N} \frac{C}{\pi n}. \tag{7}$$

Now, since

$$\lim_{A \to +\infty} \sum_{n=1}^{N} \frac{C}{\pi n} = \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{C}{\pi n} = \sum_{n=1}^{\infty} \frac{C}{\pi n} = +\infty$$

and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we have that the integral is indeed divergent.

LINEAR ALGEBRA

Problem 1

An n-dimensional vector space V and a linear transformation $\theta : V \to V$ are given. Consider the powers of θ, i.e. $1, \theta, \theta^2, \ldots$. Prove that there exists a nonzero integer s such that

$$V = \operatorname{im}(\theta^s) \oplus \ker(\theta^s).$$

SOLUTION

First, let's prove the following result:

Lemma 0.2. *If W_1 and W_2 are subspaces of a vector space V, then we have*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Proof. Let $\dim(W_1 \cap W_2) = k$ and let $\{u_1, \ldots, u_k\}$ be a basis of $W_1 \cap W_2$. Extend this basis to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ of W_1 and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ of W_2. hence, $\dim(W_1) = k + m$ and $\dim(W_2) = k + n$. Recall that

$$\begin{aligned} W_1 + W_2 =&(\{u_1, \ldots, u_k, v_1, \ldots, v_m\} \cup \{u_1, \ldots, u_k, w_1, \ldots, w_n\}) \\ =&(\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}). \end{aligned}$$

To show that $\dim(W_1+W_2) = (k+m)+(k+n)-k = k+m+n$, one needs to show that the set $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is linearly independent. For this, suppose

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_1 v_1 + \ldots + \beta_m v_m + \gamma_1 w_1 + \ldots + \gamma_n w_n = \overline{0}. \tag{8}$$

Let

$$\overline{v} = \alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_1 v_1 + \ldots + \beta_m v_m \in W_1.$$

Then

$$\overline{v} = -(\gamma_1 w_1 + \ldots + \gamma_n w_n) \in W_2.$$

So $\overline{v} \in W_1 \cap W_2$ and since this has $\{u_1, \ldots, u_k\}$ as a basis, there exists $\lambda_1, \ldots, \lambda_k$ such that

$$\overline{v} = \lambda_1 u_1 + \ldots + \lambda_k u_k.$$

Together it implies that

$$\lambda_1 u_1 + \ldots + \lambda_k u_k + \gamma_1 w_1 + \ldots + \gamma_n w_n = \overline{0},$$

and since $\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ is linearly independent, we have

$$\lambda_1 = \ldots = \lambda_k = \gamma_1 = \ldots = \gamma_n = 0.$$

Substituting this into 8 gives

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_1 v_1 + \ldots + \beta_m v_m = \overline{0},$$

and since $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ is linearly independent, we get

$$\alpha_1 = \ldots = \alpha_k = \beta_1 = \ldots = \beta_m = 0.$$

□

For the given $\theta : V \to V$, it is obvious that $\ker(\theta) \subseteq \ker(\theta^2) \subseteq \ldots$ is an ascending chain of the subspaces of V. As V is finite-dimensional, it follows that there exists an $s \in \mathbb{N}$ such that $\ker(\theta^s) = \ker(\theta^{s+1})$.

We claim that if $\ker(\theta^s) = \ker(\theta^{s+1})$, then $\ker(\theta^s) = \ker(\theta^{s+k})$ for all $k \in \mathbb{N}$. This can be proved using induction on k:

1. For $k = 1$, there is nothing to prove.

2. Suppose the assertion holds for k.

3. To prove the assertion for $k + 1$, note first that $\ker(\theta^s) \subseteq \ker(\theta^{s+k+1})$. Next, consider arbitrary $x \in \ker(\theta^{s+k+1})$. We get $\theta^{s+k}(\theta x) = 0$. That is, $\theta x \in ker(\theta^{s+k})$, which yields $\theta x \in \ker(\theta^s)$. This implies $\theta^{s+1} x = 0$ which means $x \in \ker(\theta^{s+1})$. But $\ker(\theta^{s+1}) = \ker(\theta^s)$. Thus, $x \in \ker(\theta^s)$. As $x \in \ker(\theta^{s+k+1})$ was arbitrary, we get $\ker(\theta^{s+k+1}) \subseteq \ker(\theta^s)$. Therefore, $\ker(\theta^{s+k+1}) = \ker(\theta^s)$, proving the claim.

We now show that

$$V = \operatorname{im}(\theta^s) \oplus \ker(\theta^s),$$

where s is as in the above. Bby the Rank-Nullity Theorem $\dim(\operatorname{im}(\theta^s)) + \dim(\ker(\theta^s)) = \dim V$. On the other hand, we have Lemma 0.2, which in view of the preceding equality together with the fact that both $\operatorname{im}(\theta^s)$ and $\ker(\theta^s)$ are subspaces of V, implies $V = \operatorname{im}(\theta^s) \oplus \ker(\theta^s)$ as soon as we show that

$\mathrm{im}(\theta^s) \cap ker(\theta^s) = \{0\}$. To see this, consider an arbitrary $x \in \mathrm{im}(\theta^s) \cap \ker(\theta^s)$. It follows that there exists $x_1 \in V$ such that $x = \theta^s x_1$, from which, we obtain $\theta^{s+s} x_1 = 0$ because $x \in \ker(\theta^s)$. On the other hand, by the claim we made in the above, we have $\ker(\theta^{s+s}) = \ker(\theta^s)$, yielding $x_1 \in \ker(\theta^s)$. In other words, $x = \theta^s x_1 = 0$. Since $x \in \mathrm{im}(\theta^s) \cap \ker(\theta^s)$ was arbitrary, we conclude that $\mathrm{im}(\theta^s) \cap \ker(\theta^s) = \{0\}$, which is what we want. Therefore, $V = \mathrm{im}(\theta^s) \oplus \ker(\theta^s)$, completing the proof.

Problem 2

Let $A = (a_{ij})$ be an $n \times n$ matrix over the field of real numbers such that for all i we have $\sum_{j=1}^n a_{ij} = a$. Consider a natural number $n = 2, 3, \ldots$. If $A^n = I$, find all possible values of a.

SOLUTION

Let's first consider the case with $A^2 = I$. Let $A = (a_{ij}) \in M_n(\mathbb{R})$. Let $(A^2)_{ij}$ denote the ij entry of the matrix A^2. Since $A^2 = I$, we have

$$\sum_{i=1}^n \sum_{j=1}^n (A^2)_{ij} = \sum_{i=1}^n \sum_{j=1}^n (I)_{ij} = n.$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^n (A^2)_{ij} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ik} a_{kj} = \sum_{i=1}^n \sum_{k=1}^n a_{ik} \sum_{j=1}^n a_{kj} = \\ = \sum_{i=1}^n \sum_{k=1}^n a_{ik} a = a \sum_{i=1}^n \sum_{k=1}^n a_{ik} = a \sum_{i=1}^n a = na^2\end{aligned}$$

Therefore, $na^2 = n$, yielding $a = \pm 1$. It is also worth to note that both cases indeed happen because $A = \pm I$ yields $a = \pm 1$, respectively.

Repeated application of the above procedure yields $a^n = 1$, which would give $a = 1$ if n is even, and $a = \pm 1$ is n is odd.

DISCRETE MATH

by Jozsef Solymosi

Problem 1

Let $G_n = [n] \times [n]$ denote the points of the integer grid, i.e.

$$G_n = \{(a, b) | a, b \in \mathbb{N}, 1 \leq a, b \leq n\}.$$

Prove the following statement: If we have a point set $S \subset G_n$ and

$$|S| \geq (2n)^{\frac{3}{2}}$$

then S contains five points,

$$p_1 = (a_1, b_1), p_2 = (a_2, b_2), p_3 = (a_3, b_3), p_4 = (a_4, b_4), p_5 = (a_5, b_5)$$

such that $a_1 = a_2, b_1 = b_4, a_4 = a_5, b_2 = b_3$ and $a_3 + b_3 = a_5 + b_5$.

SOLUTION

It is easier to solve the problem if we add an extra point, $q = (q_x, q_y)$ where $q_x = a_3$ and $q_y = b_1$. In this way, we are looking for three vertical point-pairs, $(p_1, p_2), (q, p_3)$ and (p_4, p_5). See the picture below.

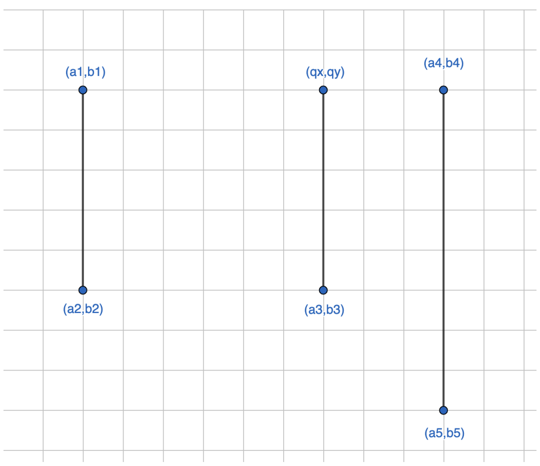

Figure 1: A possible configuration of the 5 + 1 points

We will continue to work with the vertical pairs.

$$X := \{(x_1, y_1), (x_2, y_2) \in S | x_1 = x_2, y_1 > y_2\}.$$

To estimate the size of X, we count the number of point pairs in the same column. If there are m_i points in column i (points having the same x-coordinate) then the number of pairs is $\binom{m_i}{2}$. The cardinality of X is minimal if every column has about the same number of points. We have

$$\sum_i \binom{m_i}{2} \geq n \binom{\frac{1}{n}\sum_i m_i}{2}$$

$$|X| \geq n \binom{|S|/n}{2}.$$

Now we remove the elements of X for which the number of vertical pairs with the same y-coordinates is less than three. Then, if we can find two vertical pairs, (q, p_3) and (p_4, p_5), satisfying $q_x = a_3, q_y = b_4, a_4 = a_5$, and $a_3 + b_3 = a_5 + b_5$ then there is always a choice for $(p_1, p_2) \in X$ with $a_3 \neq a_1 = a_2 \neq a_4$ such that $b_1 = b_4$ and $b_2 = b_3$ providing the five points we are looking for.

There are $\binom{n}{2}$ possible x-coordinate pairs of the elements of X. If an x-coordinate pair appears in less than three elements of X, then remove these elements from X. The remaining set is denoted by X'. Since we removed at most two elements of X from the "sparse" x-coordinate pairs, we have

$$|X'| \geq |X| - 2\binom{n}{2}.$$

Let's partition the elements of X' into partition classes, $C_{s,t}$ depending on the y-coordinate of their first point and the sum of the coordinates of their second point.

$$C_{s,t} = \{(x_1, y_1), (x_2, y_2) \in X' | y_1 = s, x_2 + y_2 = t\}.$$

There are at most $n-1$ possible values for s, and at most $2n-2$ possible values for t. (In any point pair, in the upper point $s > 1$ and for the lower point $t < 2n - 1$) The number of partition classes is at most $2(n-1)^2$. If $|X'| > 2(n-1)^2$ then we have two vertical pairs in it, (q, p_3) and (p_4, p_5), satisfying $q_x = a_3, q_y = b_4, a_4 = a_5$, and $a_3 + b_3 = a_5 + b_5$. Putting everything together, we get that if

$$n\binom{|S|/n}{2} - 2\binom{n}{2} \geq 2n(n-1) > 2(n-1)^2$$

$$\binom{|S|/n}{2} \geq 3(n-1)$$

then there are five points in S satisfying the required conditions. What is left is to check that if $|S| \geq (2n)^{\frac{3}{2}}$ then the above inequality holds.

Problem 2

Let's write 100 as the sum of 50 numbers, where the summands are integers between one and 50.

$$x_1 + x_2 + \ldots + x_{49} + x_{50} = 100 \quad x_i \in \mathbb{N}, \quad 1 \leq x_i \leq 50.$$

An $I \subset [50] = \{1, 2, 3, \ldots, 49, 50\}$ is called a *halving partition set* if the x_i-s can be partitioned into two sets, both having the same sum, 50.

$$\sum_{i \in I} x_i = \sum_{i \in \{[50] \setminus I\}} x_i.$$

Prove that there is always a halving partition set.

SOLUTION

If all x_i are the same, then there are $\binom{50}{25}$ halving partition sets. Let us suppose that $x_1 < x_2$ and consider the following 51 numbers.

$$b_1 = x_1, b_2 = x_2, b_3 = x_1 + x_2, b_4 = x_1 + x_2 + x_3, \ldots, b_{51} = \sum_{i=1}^{50} x_i = 100.$$

This is an increasing sequence. There are 51 positive integers, so at least two of them are the same modulo 50. Let us suppose that $b_i \equiv b_j \pmod{50}$ for some $i < j$ indices. This is only possible if $b_j - b_i = 50$. If $i \neq 2$, the set

$$I = \{i, i+1, \ldots, j-1\}$$

is a halving partition set with $j \geq i+1$. The $i = 2$ case needs a slightly different treatment. If $i = 2$ then

$$I = \{1, 3, \ldots, j-1\}$$

is a halving partition set.

By the definition, the complement of I is also a halving partition set.

$$I' = \overline{I} = [50] \setminus \{i, \ldots, j-1\}.$$

Remark: If one of the x_i-s is 50, then there are exactly two halving partition sets. There are different possible arguments giving the same answer.