UBC Math Circle 2018 Problem Set 2

I. Introductory Problems

1. A group of 100 students are gathered in a hallway with 100 lockers, all of which are shut and unlocked. The first student goes along the hallway and opens every locker. The second student goes along the hallway closes every second locker. The third student goes along the hallway and, at every third locker, closes the locker if it is open and opens the locker if it is closed. The students proceed in this way until all 100 students have gone. How many lockers remain open?

2. Devise a game of sprouts that starts with 5 nodes and ends after 10 turns and then form a lower bound on the number of turns in any size game.

3. Eugene is playing a game on an $n \times n$ chessboard. The game begins with a white queen on $(1, 1)$ and a black queen on $(1, n)$. All other squares have green pawns in them. In a move, a player must capture a piece with their queen (either a green pawn or the enemy queen). If a player's queen is captured or if they cannot capture any pieces on their move, they lose. If white moves first, who wins assuming optimal play?

II. Intermediate Problems

4. (IMO 2016 Shortlisted Problem) The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leaders in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

5. Foster and Tom play a game with 3 piles of stones, moving alternately. Initially the 3 piles have a, b, and c stones. During a single move, the player can do exactly one of the following

   1. Choose any non-empty pile and take one stone from it.
   2. Choose any two non-empty piles and take one stone from each.
   3. Move one stone from a pile to another

   The player who takes the last stone wins. Foster moves first, who will win? (May depend on what a, b, and c are)

6. Foster and Tom play a game of colouring square regions on a 43×2018 rectangular grid. Initially, each square in the rectangular grid is uncoloured. Each turn consists of a player choosing a square region whose sides are parallel to the edges of the grid, and whose interior does not contain any coloured squares. The player who colours the last uncoloured square wins. They alternate turns, with Foster going first. Who will win?

7. n people from math circle are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on. Players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

III. Advanced Problems

8. A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking either

    1. one bean from a heap, provided at least two beans are left behind in that heap, or
    2. a complete heap of two or three beans.

    The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy

9. Little Zeno the rabbit is standing at the foot of the real number line. He knows that making a jump from any real a to b, $b > a$, costs exactly $b^3 - ab^2$ jump dollars. For what cost values, $c \in \mathbb{R}$, can Little Zeno traverse from 0 to 1 in a finite number of jumps with total jump dollar cost exactly c?