UBC Math Circle 2018 Problem Set 3

I. Introductory Problems

1. Let MNP be a triangle with vertices $M(1, 4)$, $N(5, 3)$, and $P(5, c)$. Determine the sum of the two values of c for which the area of MNP is 14.

> **Solution:** (Stolen from Euclid 2015) Points $N(5, 3)$ and $P(5, c)$ lie on the same vertical line. We can consider NP as the base of MNP. Suppose that the length of this base is b. The corresponding height of MNP is the distance from $M(1, 4)$ to the line through N and P. Since M lies on the vertical line $x = 1$ and N and P lie on the vertical line $x = 5$, then the height is $h = 4$. Since the area of MNP is 14, then $\frac{1}{2}bh = 14$. Since $h = 4$, then $\frac{1}{2}b(4) = 14$ or $2b = 14$ and so $b = 7$. Therefore, $P(5, c)$ is a distance of 7 units away from $N(5, 3)$. Since NP is a vertical line segment, then $c = 3 + 7$ or $c = 37$, and so $c = 10$ or $c = 4$. The sum of these two values is $10 + (4) = 6$. (We could also have noted that, since the two values of c will be symmetric about $y = 3$, then the average of their values is 3 and so the sum of their values is $2 \cdot 3 = 6$.)

2. For some positive integers k, the parabola with equation $y = \frac{x^2}{k}5$ intersects the circle with equation $x^2 + y^2 = 25$ at exactly three distinct points A, B and C. Determine all such positive integers k for which the area of triangle ABC is an integer.

> **Solution:** (also stolen from Euclid 2015) First, we note that since k is a positive integer, then $k1$. Next, we note that the given parabola passes through the point (0, 5) as does the given circle. (This is because if $x = 0$, then $y = \frac{0^2}{k}5 = 5$ and if $(x, y) = (0, 5)$, then $x^2 + y^2 = 0^2 + (5)^2 = 25$, so (0, 5) satisfies each of the equations.) Therefore, for every positive integer k, the two graphs intersect in at least one point. If $y = 5$, then $x^2 + (5)^2 = 25$ and so $x^2 = 0$ or $x = 0$. In other words, there is one point on both parabola and circle with $y = 5$, namely (0, 5). Now, the given circle with equation $x^2 + y^2 = 25 = 5^2$ has centre (0, 0) and radius 5. This means that the y-coordinates of points on this circle satisfy $5y5$. To find the other points of intersection, we re-write $y = \frac{x^2}{k}5$ as $ky = x^2 5k$ or $x^2 = ky + 5k$ and substitute into $x^2 + y^2 = 25$ to obtain $(y + 5)(y + (k - 5)) = 0$ and so $y = -5$ or $y = 5 - k$. (We note that since the two graphs intersect at $y = 5$, then $(y + 5)$ was going to be a factor of the quadratic equation $y^2 + ky + (5k25) = 0$. If we had not seen this, we could have used the quadratic formula.) Therefore, for $y = 5k$ to give points on the circle, we need $55k$ and $5k5$. This gives $k10$ and $k0$. Since k is a positive integer, the possible values of k to this point are $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. If $k = 1$, then $y = 51 = 4$. In this case, $x^2 + 4^2 = 25$ or $x^2 = 9$ and so $x = 3$. This gives the two points (3, 4) and (3, 4) which lie on the parabola and circle. Consider the

three points $A(3,4), B(3,4)$ and $C(0,5)$. Now AB is horizontal with $AB = 3(3) = 6$. (This is the difference in x-coordinates.) The vertical distance from AB to C is $4(5) = 9$. (This is the difference in y-coordinates.) Therefore, the area of triangle ABC is $\frac{1}{2}(6)(9) = 27$, which is a positive integer. We now repeat the calculations for $k = 1,2,3,4,5,6,7,8,9,10$. When $k = 10$, we have $y = 5k = 5$ and $x = 0$ only, so there is only one point of intersection. Finally, the values of k for which there are three points of intersection and for which the area of the resulting triangle is a positive integer are $k = 1,2,5,8,9$.

II. Intermediate Problems

3. Let P be a point in the interior of $\triangle ABC$. Let $[XYZ]$ denote the area of $\triangle XYZ$. Let lines AP, BP, and CP intersect BC, CA, and AB at D, E, and F respectively. Prove that $[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$ if and only if P lies on at least one of the medians of $\triangle ABC$.

Solution: We can use barycentric coordinates.

Let $P = (\alpha : \beta : \gamma)$. Then $D = (0 : \beta : \gamma)$, $E = (\alpha : 0 : \gamma)$, and $F = (\alpha : \beta : 0)$. Then

$$[PAF] + [PBD] + [PCE] =$$
$$\frac{[ABC]}{\alpha+\beta} \det \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & 0 \\ \alpha & \beta & \gamma \end{bmatrix} + \frac{[ABC]}{\beta+\gamma} \det \begin{bmatrix} 0 & 1 & 0 \\ 0 & \beta & \gamma \\ \alpha & \beta & \gamma \end{bmatrix} + \frac{[ABC]}{\gamma+\alpha} \det \begin{bmatrix} 0 & 0 & 1 \\ \alpha & 0 & \gamma \\ \alpha & \beta & \gamma \end{bmatrix}$$

We can equate this with $\frac{1}{2}[ABC]$ to get

$$\frac{\alpha\beta}{\gamma+\alpha} + \frac{\beta\gamma}{\alpha+\beta} + \frac{\gamma\alpha}{\beta+\gamma} = \frac{1}{2}.$$

Now, if we plug in $\alpha = \beta$, we see that the two sides are indeed equal. It suffices to show that the above equation is equivalent to the following

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = (\alpha-\beta)(1-\alpha-2\beta)(1-2\alpha-\beta) = 0.$$

Pray really hard.

4. (APMO 2015) Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point Z distinct from B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $AB = VW$.

Solution: It suffices to show that $\measuredangle BCA = \measuredangle VZW$.

Miquel's quadrilateral Theorem tells us that the circumcircles of the four triangles in a complete quadrilateral all meet at one point (the Miquel point). In this problem, consider the complete quadrilateral consisting of the lines AB, BC, CA, and XY. Since the circumcircle of $\triangle BXD$ meets ω at Z, and the circumcircle of $\triangle AXY$ does not intersect B, the point Z must be the Miquel point. Hence Z lies on the circumcircle of $\triangle CDY$. It follows that (using directed angles modulo 180°)

$$\measuredangle VZW = \measuredangle VZY = \measuredangle DZY = \measuredangle DCY = \measuredangle BCY = \measuredangle BCA.$$

III. Advanced Problems

5. Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC respectively, such that $AX = AY = 1$. Segments BC and XY intersect at M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

Solution: Without loss of generality, let $AC > AY$ and $AX < AB$.

Recall that the length of the tangent from a point to its corresponding excircle is equal to the semiperimeter. Let U and V lie on rays AB and AC respectively, such that $AU = AV = 2$. Then, the A-excircle (denote this α) is tangent to AB and AC at U and V respectively. Let α be tangent to BC at T.

Let's construct a circle ω with zero radius at A. Then, X and Y lie on the radical axis of α and ω. Hence M also lies on the radical axis, and $AM = MT$.

It follows that $AB + BM + AM = AB + BM + MT = AB + BT = AB + BU = 2$

6. (USAMO 2009) Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in a circle ω, and point G lines inside the triangle BCD. Rays AG and BG meet ω again at points P and Q respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

   Challenge/Hint/Trap: Solve this problem using inversion (of course, you could also solve it without using inversion).

Solution: There are six points on ω, so let's use inversion to transform them into collinear points. This means we want to invert around a point on ω, let's choose B because it has a lot of lines passing through it. The radius of inversion is arbitrary.

- The cyclic trapezoid $ABCD$ is transformed into the point B, and three collinear points A', C', and D'. Since $\overline{AB} \parallel \overline{CD}$, we have $\overline{A'B}$ tangent to the circumcircle of $\triangle BC'D'$.
- The point G is transformed into some point G' outside of $\triangle BC'D'$.
- The point P is transformed to the intersection of the line $\overline{A'C'}$ and the circumcircle of $\triangle BA'G'$.
- The point Q is transformed to the intersection of the lines $\overline{A'C'}$ and $\overline{BG'}$.
- The points R and S are transformed onto the circle through B and G' tangent to $\overline{A'B}$. Moreover, R' lies on the line $\overline{BD'}$ and S' lies on the line $\overline{BC'}$.

We want to show that $P'Q'R'S'$ is cyclic if and only if $\overline{BG'}$ bisects $\angle C'BD' = \angle R'BS'$.

Observe that $\overline{C'D'}$ and $\overline{R'S'}$ remain parallel because there is a dilation that takes the circumcircle of $BC'D'$ to the circumcircle of $BR'S'$. This means that $P'Q'R'S'$ is a trapezoid with $\overline{P'Q'} \parallel \overline{R'S'}$, hence it suffices to prove $P'S' = Q'R'$ if and only if $\overline{BG'}$ bisects $\angle R'BS'$.

Let $\overline{P'G'}$ meet the circumcircle of $BR'S'$ again at X. Then

$$\measuredangle Q'P'G' = \measuredangle A'P'G' = \measuredangle A'BG' = \measuredangle BXG',$$

so $\overline{BX} \parallel \overline{R'S'}$, hence $BR'S'X$ is an isoceles trapezoid.

Observe that G' is the midpoint of arc $R'S'$ if and only if $\overline{BG'}$ bisects $\angle R'BS'$. If G' is the midpoint of arc $R'S'$, then by symmetry, $P'Q'R'S'$ is also isoceles, hence cyclic. If $P'Q'R'S'$ is cyclic, then it must be isoceles, whence by symmetry G' is the midpoint of arc $R'S'$.

7. A convex and bounded set C in $\mathbb{R}^3$ has boundary $\partial C = B$. Prove that the set of points $D = B + B = \{p + q \mid p, q \in B\}$ is also a convex set. A set C is called convex if for any two points $x, y \in C$, then $tx + (1-t)y \in C$ for any $t \in (0, 1)$.

Solution: Consider the problem in $\mathbb{R}^2$. If its true in $\mathbb{R}^2$, then there is natural extension to $\mathbb{R}^3$. Observe that any point in D can be written as twice a point in C, so we need to show that for any point $x \in C$ there exists $p, q \in C$ such that $(p+q)/2 = x$. Consider if we rotated a line around x, this let's construct a continuous function that returned the difference between $|p - x|$ and $|q - x|$. Since rotating the line by $180°$ switches the role of p and q, we can apply the intermediate value theorem to find that there exists a pair of points p and q colinear with x such that $|p - x| = |q - x|$, so we are done.