UBC Math Circle 2018 Problem Set 5

I. Introductory Problems

1. How many different ways can you arrange the letters of the word PEPPER?

Solution: There are 3 P's, 2 E's and 1 R so we have 6! possible ways to arrange the letters and 3!2! ways to permute the non-distinct letters in each arrangement. Therefore we have

$$\frac{6!}{3!2!} = 60$$

different arrangements.

2. If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two are black?

Solution: We have 11(10)(9) possible outcomes drawing three balls from the bowl. There are 6(5)(4) in which the first ball is black and the other two are white; 5(6)(4) in which the first and last balls are black and the second ball is white; 5(4)(6) outcomes where the last ball is white and the first two are black. Hence the probability of drawing a white ball and two black balls is:

$$\frac{6(5)(4) + 5(6)(4) + 5(4)(6)}{11(10)(9)} = \frac{3(120)}{990} = \frac{4}{11}$$

II. Intermediate Problems

3. In the card game bridge the entire 52 card deck is dealt out between 4 players. What is the probability that one player is dealt all 13 spades?

Solution: The probability of any one player being dealt the spades is $\frac{1}{\binom{52}{13}}$. As there are four players and these events are mutually exlcusive we therefore have probability:

$$\frac{1}{\binom{52}{13}} + \frac{1}{\binom{52}{13}} + \frac{1}{\binom{52}{13}} + \frac{1}{\binom{52}{13}} = \frac{4}{\binom{52}{13}} \approx 6.3(10)^{-12}$$

4. A laboratory test is 95 percent effective in detecting a certain disease when it is, in fact, present. However the test also yields a ”false positive” result for 1 percent of the healthy persons tested. If 0.5 percent of the population actually has the disease what is the probability that a person has the disease given that the test result is positive?

Solution: Since 0.5 percent of the population actually has the disease it follows that, on average, 1 person out of every 200 tested positive for it. The test will correctly confirm that this person has the disease with probability 0.95. Thus, on average, out of every 200 persons tested, the test will correctly confirm that person has the disease. On the other hand out of the (on average) 199 healthy people, the test will inaccurately state that 0.1(199) of these people have the disease. Hence for every diseased person the test correctly states is ill there are (on average) 0.1(199) healthy persons who the test incorrectly states are ill. Thus, the proportion of time that the test result is correct when it states a person is ill is:

$$\frac{0.95}{0.95 + 0.1(199)} = \frac{95}{294} \approx 0.323$$

This is easier to show using Baye's formula: Let D be the event that the person has the disease and E the event that the test result is positive. Then the desired probability is:

$$\begin{aligned} P(D|E) &= \frac{P(DE)}{P(E)} \\ &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \\ &= \frac{0.95(0.005)}{0.95(0.005) + 0.1(0.995)} \\ &= \frac{95}{294} \\ &\approx 0.323 \end{aligned}$$

5. Three real numbers a, b, c are selected uniformly at random from the interval $[0, 1]$. What is the probability that there is a triangle with side lengths a, b, c?

Solution: Without loss of generality, suppose, $a \leq b \leq c$. Further, suppose $c = 1$, since we can rescale a and b appropriately if $c \neq 1$. The question then becomes (using the triangle inequality) “If a, b are selected uniformly at random from the interval $[0, 1]$, what is the probability that $a + b > 1$?”, which has probability $\frac{1}{2}$.

III. Advanced Problems

6. Let S be a finite set of points in the plane such that no three points are collinear. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P and $b(P)$ be the number of points of S that are outside of P. Prove that for every $x \in (0,1)$, we have

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

Note: In this problem, we consider a line segment, a single point, and the empty set to be convex polygons of 2, 1, and 0 points respectively.

Solution: Randomly colour the points red with probability x and blue with probability $1-x$. For each polygon P, let E_P be the event that all vertices on the perimeter of P are red, and all points in S outside of P are blue. The left hand side of the equation is the probability that E_P holds for some P. We see that E_P holds for the convex hull of the black points, which is a valid convex polygon P, and the result follows.

7. Let k be a positive integer. Suppose the integers 1, 2, 3, ... , $3k+1$ are written down in random order. What is the probability that at no time in the process the sum of all integers currently written down is a positive integer divisible by 3?

Solution: Assume that we have an ordering of $1, 2, \ldots, 3k+1$ such that no initial subsequence sums to 0 mod 3. If we omit the multiples of 3 from this ordering, then the remaining sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$ or $-1, -1, 1, -1, 1, \ldots$. Since there is one more integer in the ordering congruent to 1 mod 3 than to -1, the sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$.

It follows that the ordering satisfies the given condition if and only if the following two conditions hold: the first element in the ordering is not divisible by 3, and the sequence mod 3 (ignoring zeroes) is of the form $1, 1, -1, 1, -1, \ldots$. The two conditions are independent, and the probability of the first is $(2k+1)/(3k+1)$ while the probability of the second is $1/\binom{2k+1}{k}$, since there are $\binom{2k+1}{k}$ ways to order $(k+1)$ 1's and k -1's. Hence the desired probability is the product of these two, or $\frac{k!(k+1)!}{(3k+1)(2k)!}$.