UBC Math Circle 2019 Problem Set 1

Theme: Induction

I. Introductory Problems

1. Prove that
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

> **Solution:** We use induction on n.
>
> The base case is $n=2$. Observe that $\left(1-\frac{1}{4}\right)=\frac{3}{2\cdot 2}$, so the closed form holds for $n=2$.
>
> Let $n \geq 2$ be arbitrary, and suppose that the closed form holds for n. We want to show that the closed form also holds for $n+1$.
> $$\begin{aligned}\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) &= \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{(n+1)^2}\right)\\ &= \frac{(n+1)^2-1}{2n(n+1)}=\frac{n+2}{2(n+1)}.\end{aligned}$$
>
> This means that the closed form holds for $n+1$ if it holds for n, so we are done by induction.

2. Prove that for any $n \geq 0$, if we remove any square from a 2^n x 2^n chessboard, the remainder can be tiled with "L" shaped pieces (more precisely, a 2x2 square with one of the corners missing)

> **Solution:** Base case: n=0. Then we have a 1x1 chessboard, and removing a square leaves us with an empty grid, which can vacuously be tiled with L shaped pieces.
>
> Suppose true up to n. Consider a 2^{n+1} x 2^{n+1} chessboard. Divide the chessboard into 4 quadrants, each of which is a smaller 2^n x 2^n chessboard. The removed square lies in one of these quadrants. Place an L shaped piece in the centre of all 4 quadrants such that it intersects the 3 quadrants without the square removed. Apply the induction hypothesis to each quadrant.

II. Intermediate Problems

3. Consider n lines in a plane. These n lines divide the plane into some regions (there are some infinite regions). We say two regions are adjacent if their borders share some line

segment (they are not adjacent if their borders only share a point). Prove that it is possible to colour each region either 1 or 2 so that no region coloured 1 is adjacent to a region that is coloured 2 (yes, 1 and 2 are colours).

Solution: We use induction on n.

The base case is $n = 0$. The plane is divided into one region, and we can just colour it any colour.

Suppose that the statement is true for some arbitrary $n \geq 0$. We want to show that the statement also holds for $n + 1$. Consider a valid colouring of the regions after adding the first n lines. Now we add the last line. Note that if we flip the colours on one side of the line, then we will get a valid colouring.

4. Prove that for all positive integers n,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \sqrt{\frac{1}{3n}}$$

Solution: Use the inductive hypothesis that the product $\leq \sqrt{\frac{1}{3n+1}}$, instead of the obvious one.

5. There are $2n$ points around a circle, n of which are blue, and the other n are red. Going counter-clockwise, we keep a count of how many red and blue points we have passed. If at all times, the number of red points as at least the number of blue points, we say the trip is *good*. Prove that no matter how we colour the $2n$ points, we can start somewhere so that we have a *good* trip.

Solution: We use induction on n.

The base case is $n = 0$. Observe that $n = 0$ is trivial. There are no points, so we can't get more blue than red.

Let $n \geq 0$ be arbitrary, we want to show that if the there exists a *good* trip for n, then there also exists a *good* trip for $n + 1$. Consider the $2(n + 1)$ points around the circle, of which $n+1$ are red and the other $n+1$ are blue. Since we have two different colours, going counter-clockwise, we can find some place where we see a red point with a blue point immediately following the red one. Suppose we remove these two points from the $2(n + 1)$ points, then we have $2n$ points that satisfy the conditions, so there exists a *good* trip for these $2n$ points.

Now, I claim that if we take the *good* trip and add these two points, we still get a *good* trip because the red one comes before the blue one in the trip (Exercise). This means we also have a *good* trip for $2(n+1)$ points.

Induction completes the proof.

6. Prove that for all n

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Solution: Let $n = 2$:

$$\begin{aligned}
LHS^2 - RHS^2 &= (a_1b_1 + a_2b_2)^2 - (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\
&= (a_1b_1)^2 + (a_2b_2)^2 + 2a_1a_2b_1b_2 - (a_1a_2)^2 - (b_1b_2)^2 - (a_1b_2)^2 - (a_2b_1)^2 \\
&= -[(a_1b_2)^2 - 2a_1b_2a_2b_1 + (a_2b_1)^2] \\
&= -[(a_1b_2) - (a_2b_1)]^2 \\
&\leq 0
\end{aligned}$$

Assume that it holds for $[2, n]$. For $n+1$, if we let

$$\begin{aligned}
LHS &= |\sum_{i=1}^{n+1} a_i b_i| \\
&\leq |\sum_{i=1}^{n} a_i b_i| + |a_{n+1}b_{n+1}| \\
&\leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + |a_{n+1}b_{n+1}| \\
&\leq (\sqrt{\sum_{i=1}^{n} a_i^2}^{\,2} + |a_{n+1}|^2)^{\frac{1}{2}}(\sqrt{\sum_{i=1}^{n} b_i^2}^{\,2} + |b_{n+1}|^2)^{\frac{1}{2}} \\
&= \sqrt{\sum_{i=1}^{n+1} a_i^2}\sqrt{\sum_{i=1}^{n+1} b_i^2}
\end{aligned}$$

III. Advanced Problems

7. A sphere is covered with $n \geq 4$ hemispheres. Prove that we can choose 4 of these hemispheres such that the sphere is still covered after removing all other hemispheres.

Solution: Let $\{x_i\}$ be the set of "peak" points of the hemispheres. Consider the convex hull of these points, which is a convex polyhedron. Then I claim that the hemispheres cover the sphere if and only iff the polyhedron contains the centre of the the sphere.

The given hemispheres don't cover the sphere iff there is a point, y, that is not covered iff there are no x_i within a hemisphere of y iff all the x_i are contained in some open hemisphere iff the convex polyhedron don't contain the centre.

Now take a tetrahedral decomposition of the given convex polyehdron to find a tetrahedron containing the centre, and hence 4 hemispheres covering the sphere.

8. (APMO 1999) Let $\{a_i\}$ be a sequence of real numbers that satisfy $a_{i+j} \leq a_i + a_j$ for all $i, j \geq 1$ (not necessarily distinct). Prove that

$$\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_{2019}}{2019} \geq a_{2019}.$$

Solution: Proceeding by strong induction, we will show for all $n \geq 1$ (including $n = 2019$) that

$$\sum_{i=1}^{n} \frac{a_i}{i} \geq a_n$$

The base case $n = 1$ holds trivially.

Inductive step: assuming the statement holds for all $1 \leq n \leq k$, we will prove it for $n = k + 1$.

By the strong inductive hypothesis,

$$\sum_{j=1}^{k} (\sum_{i=1}^{j} \frac{a_i}{i}) \geq \sum_{j=1}^{k} a_j$$

$$\sum_{j=1}^{k} (k+1-j)\frac{a_j}{j} \geq \sum_{j=1}^{k} a_j$$

$$(k+1)\sum_{j=1}^{k} \frac{a_j}{j} \geq 2\sum_{j=1}^{k} a_j = \sum_{j=1}^{k} (a_j + a_{k+1-j}) \geq ka_{k+1}$$

Rearranging, we obtain the desired inequality:

$$\sum_{j=1}^{k+1} \frac{a_j}{j} \geq a_{k+1}$$

9. Prove that given $a, x_1, ...x_n \geq 0$

$$\prod_{i=1}^{n}(a+x_i) \geq \left(a+\prod_{i=1}^{n} x_i^{1/n}\right)^n$$

Solution: For n = 1 we have equality. We shall proceed by Cauchy induction.

Suppose true for $n = k$. Define $m_j = \prod_{i=1}^{j} x_i^{1/j}$. We want to show it holds for $n = 2k$, as well as $n = k - 1$.

First, let $n = 2k$ and split up the x_i into two groups: $1 \leq i \leq k$ and $k+1 \leq i \leq 2k$. For each group, applying the induction hypothesis nets us the following:

$$\prod_{i=1}^{k}(a+x_i) \geq (a+m_k)^k \tag{1}$$

$$\prod_{i=k+1}^{2k}(a+x_i) \geq \left(a+\prod_{i=k+1}^{2k} x_i^{1/k}\right)^k \tag{2}$$

For (2), we have

$$\left(a+\prod_{i=k+1}^{2k} x_i^{1/k}\right)^k = \left(a+\frac{m_k}{m_k}\prod_{i=k+1}^{2k} x_i^{1/k}\right)^k = \left(a+\frac{m_{2k}^2}{m_k}\right)^k \tag{3}$$

Now combining (1) and (3) we have

$$\begin{aligned}\prod_{i=1}^{2k}(a+x_i) \geq (a+m_k)^k\left(a+\frac{m_{2k}^2}{m_k}\right)^k &= \left(a^2+m_{2k}^2+a\left(m_k+\frac{m_{2k}^2}{m_k}\right)\right)^k \\ &= \left(a^2+m_{2k}^2+a\left(\left(\sqrt{m_k}-\frac{m_{2k}}{\sqrt{m_k}}\right)^2+2m_{2k}\right)\right)^k \\ &\geq \left(a^2+m_{2k}^2+2am_{2k}\right)^k = (a+m_{2k})^{2k}\end{aligned}$$

Now we want to show that it also holds for $n = k - 1$. In particular, it should hold when $x_k = m_{k-1}$. In that case, we have

$$\prod_{i=1}^{k}(a+x_i) = (a+m_{k-1})\prod_{i=1}^{k-1}(a+x_i) \geq \left(a+\prod_{i=1}^{k} x_i^{1/k}\right)^k = (a+m_{k-1})^k$$

Divide by $(a + m_{k-1})$ and we have our result.