UBC Math Circle 2019 Problem Set 10

Problems review the past 9 problems sets, in no particular order

1. Prove that if p and $p^2 + 2$ are prime, then $p^3 + 2$ is also prime.

Solution: If $p = 2$, then $p^2 + 2 = 6$, which is not prime.

If $p = 3$, then $p^2 + 2 = 11$, and $p^3 + 2 = 29$ are all prime.

Otherwise, if $p > 3$ is a prime, then $p \equiv \pm 1 \pmod 3$. Thus $p^2 \equiv 1 \pmod 3$, and $p^2 + 2 \equiv 0 \pmod 3$, so $3 < p < p^2 + 2$, and $3 \mid p^2 + 2$, so $p^2 + 2$ is not prime.

Thus the only case that works is $p = 3$.

2. Let M be a set composed of n elements and let $\mathcal{P}(M)$ be its power set. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$ that have the properties

   (a) $f(A) \neq 0$, for $A \neq \phi$;

   (b) $f(A \cup B) = f(A \cap B) + f(A \Delta B)$, for all $A, B \in \mathcal{P}(M)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Solution: I claim that $f(A) = |A|$, for all $A \in \mathcal{P}(M)$. Clearly this function works; we must now show that it is the only function with the two given properties. We shall do this by proving that any such function f that satisfies both properties must be $f(A) = |A|$.

We have $f(\emptyset) = f(\emptyset) + f(\emptyset)$, hence $f(\emptyset) = 0$. Also, if $A \subset B \subset M$, then $f(B) = f(A) + f(B - A)$ since $B = B \cup A$, $A = B \cap A$, and $B - A = B \Delta A$.

Let

$$A_0 = \emptyset \subsetneq A_1 \subsetneq \cdots \subsetneq A_{n-1} \subsetneq A_n = M$$

be an increasing sequence of subsets of M, such that $A_i = A_{i-1} \cup \{a_i\}$ for all $i = 1, \ldots, n$. Then $|A_i| = i$ for all i.

Note that $f(a_i) \geq 1$ for any i, from property (a). Hence

$$f(A_0) = 0 < f(A_1) < \cdots < f(A_{n-1}) < f(A_n) \leq n \, ,$$

which implies $f(A_i) = i$ for all i.

3. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z

respectively, and y is negative, you may replace x, y, z by $x+y, -y, z+y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution: Label the vertices $x_1, ...x_5$. Consider the invariant $S = x_1+x_2+x_3+x_4+x_5 > 0$. It is clear that this remains constant when applying a move. Now consider the function

$$f(x_1, ..., x_5) = (x_1-x_3)^2 + (x_2-x_4)^2 + (x_3-x_5)^2 + (x_4-x_1)^2 + (x_5-x_2)^2$$

It is clear that this is bounded below by 0. WLOG, apply the move to x_2, which is negative. We have that $x_1 \mapsto x_1 + x_2$, $x_2 \mapsto -x_2$, and $x_3 \mapsto x_3 + x_2$. Then our invariant becomes

$$f(x_1', ..., x_5') = (x_1-x_3)^2 + (-x_2-x_4)^2 + (x_3+x_2-x_5)^2 + (x_4-x_1-x_2)^2 + (x_5+x_2)^2$$

Expanding out the terms, we get that

$$\begin{aligned} f(x_1', ..., x_5') &= (x_1-x_3)^2 + (x_2-x_4)^2 + 4x_2x_4 + (x_3-x_5)^2 + x_2(2x_3 - 2x_5 + x_2) \\ &\quad + (x_4-x_1)^2 + x_2(2x_1 - 2x_4 + x_2) + (x_5-x_2)^2 + 4x_2x_5 \\ &= f(x_1, ..., x_5) + x_2(4x_4 + 2x_3 - 2x_5 + x_2 + 2x_1 - 2x_4 + x_2 + 5x_5) \\ &= f(x_1, ..., x_5) + 2Sx_2 < f(x_1, ..., x_5) \end{aligned}$$

as $S > 0$ and $x_2 < 0$. Thus, this function we defined takes on integer values, is strictly decreasing, and bounded below by 0, so we see that the process must terminate.

4. Prove that from any set of 2019 whole numbers, one can choose either one number which is divisible by 2019, or several numbers whose sum is divisible by 2019.

Solution: Label the numbers in the set $x_1, \ldots, x_{2019}$, consider the 2019 subsets $\{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, \ldots, x_{2019}\}$ and for each of these subsets, compute its sum. If none of these sums are divisible by 2019, then there are 2019 sums and 2018 residue classes mod 2019 (excluding 0). Therefore two of these sums are the same mod 2019, say $x_1 + \cdots + x_i \equiv x_1 + \cdots + x_j \pmod{2019}$ (with $i < j$). Then $x_{i+1} + \cdots + x_j \equiv 0 \pmod{2019}$, and the subset $\{x_{i+1}, \ldots, x_j\}$ suffices.

5. Find all polynomials such that $P(x^2+1) = P(x)^2 + 1$.

Solution: First we find the constant solutions, which are solutions to $c = c^2 + 1$. Henceforth, we assume that $\deg(P) \geq 1$.

We have

$$P(x)^2 - P(-x)^2 = (P(x^2+1) - 1) - (P((-x)^2+1) - 1) = 0$$

At least one of $P(x) - P(-x)$ and $P(x) + P(-x)$ must have infinitely many zeros and therefore must be identical to 0. Thus P is either odd or even.

If $P(x)$ is odd, then note that $P(0) = 0$. Then $P(1) = 1$, $P(2) = 2$, $P(5) = 5$, etc. and by induction we can get infinitely many values a such that $P(a) = a$. Then the polynomial $P(x) - x$ has infinitely many roots, so we see that $P(x) = x$ for all x.

Now suppose P is even. This means $P(x) = Q(x^2)$ for some polynomial Q. Let $R(x) = x^2 + 1$. There clearly exists a polynomial $S(x)$ such that $P(x) = S(x^2+1)$, just shift Q. Now by definition, $P \circ R = R \circ P$. Hence $S \circ R \circ R = R \circ S \circ R \implies S \circ R = R \circ S$, so S is a solution to the functional equation, and the degree of S is half the degree of P. By induction, we must have that $P = T \circ R \circ R ... \circ R$, where T is odd and satisfies the functional equation. However, we know the only such solution is $T(x) = x$, so $P = R \circ ... \circ R$.

6. The SOS Game is played on a 1×2020 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Solution: Label the squares 1 through 2020. The first player writes a letter in any square k. The second player now selects any empty square l such that $\min(|l - 1|, |l - 2020|, |l - k|) > 3$, and fills in a S. The first player may again play anywhere, say m. If the second player can now complete SOS, he does so; otherwise, if $l > m$ he fills an S in $l + 3$ and if $l < m$ he fills an S in $l - 3$.

We thus have two Ss separated by two empty squares. Note that if any player plays any letter between the two Ss, the other player can always complete a SOS.

Now the second player repeats the following algorithm:

If he can complete SOS on his turn, he does so. Otherwise, since there are an odd number of empty squares at the beginning of his turn, there must be either one empty square surrounded by two filled squares or one empty square surrounded by two empty squares. In both cases, the player places an O in that square. In both situations, it is easy to verify that the first player cannot win on the next turn. Eventually, assuming the second player has not won yet (and since the first player does not have a winning opportunity yet), this algorithm must stop. It follows that all of the empty squares are now in pairs. Since there are an even number of empty squares, it is the first player's turn. If the first player plays on one of any pair of empty squares that does not have Ss on both its sides, then the second player plays any letter in the other square of the pair. Then, we will only have pairs of empty squares with Ss on both sides remaining (since from the first moves we have constructed at least one, and no one has played between them otherwise the game will have ended), and the first player will have to play between them, so that the second player will win.

7. $ABCD$ and $A'B'C'D'$ are square maps of the same region, drawn to different scales and superimposed as shown in the figure. Prove that there is a unique point O on the small map that lies directly over point O' of the large map such that O and O' each represent the same place of the region.

mathcirclePS10geo.png

Solution: The point is obviously unique, because the two maps have different scales (but if P and Q where two fixed points the distance between them would be the same on both maps).

Let the small map square be ABCD and the large be A'B'C'D', where P and P' are corresponding points on the two maps. We deal first with the special case where A'B' is parallel to AB. In this case let AA' and BB' meet at O. Then triangles OAB and OA'B' are similar, so O must represent the same point. So assume A'B' is not parallel to AB.

Let the lines A'B' and AB meet at W', the lines B'C' and BC meet at X', the lines C'D' and CD meet at Y', and the lines D'A' and DA meet at Z'. We claim that the segments W'Y' and X'Z' meet at a point O inside the smaller square. W' cannot lie between A and B (or one of the vertices A, B of the smaller square would lie outside the larger square). If it lies on the same side of A to B, then Y' must lie on the same side of C to D. Thus the segment W'Y' must cut the side AD at some point Z, and the side BC at some point X. The same conclusion holds if W lies on the opposite side of A to B, because then Y must lie on the opposite side of C to D.

Similarly, the segment X'Z' must cut the side AB at some point W and the side CD at some point Y. But now the segments XZ and WY join pairs of points on opposite sides of the small square and so they must meet at some point O inside the small square.

Now the triangles WOW' and YOY' are similar (WW' and YY' are parallel). Hence OW/OY = OW'/OY'. So if we set up coordinate systems with A'B' as the x'-axis and A'D' as the y'-axis (for the large square) and AB as the x-axis and AD as the y-axis (for the small square) so that corresponding points have the same coordinates, then the y coordinate of O equals the y' coordinate of O. Similarly, XOX' and ZOZ' are similar, so OX/OZ = OX'/OZ', so the x-coordinate of O equals its x'-coordinate. In other words, O represents the same point on both maps.

An alternate solution uses the center of spiral symmetry used to go from ABCD to A'B'C'D', which stays fixed under the spiral symmetry.

8. Let $a_1, a_b, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that

$$|a_i - a_j| \geq \frac{1}{i+j} \text{ for all } i,\ j \text{ with } i \neq j.$$

Show that $c \geq 1$

Solution: For some fixed value of n, let σ be the permutation of the first n natural numbers such that $a_{\sigma(1)}, \ldots a_{\sigma(n)}$ is an increasing sequence. Then we have

$$\begin{aligned} a_{\sigma(n)} - a_{\sigma(1)} &= \sum_{i=1}^{n-1} \left| a_{\sigma(i+1)} - a_{\sigma(i)} \right| \\ &\geq \sum_{i=1}^{n-1} \frac{1}{\sigma(i+1) + \sigma(i)} \end{aligned}$$

Now, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^{n-1} \frac{1}{\sigma(i+1) + \sigma(i)} \right) \left(\sum_{i=1}^{n-1} \sigma(i+1) + \sigma(i) \right) &\geq (n-1)^2 \\ \sum_{i=1}^{n-1} \frac{1}{\sigma(i+1) + \sigma(i)} &\geq \frac{(n-1)^2}{2\sum_{i=1}^{n-1}(\sigma_i) - \sigma(1) - \sigma(n)} \\ &\geq \frac{(n-1)^2}{n(n+1) - 3} \\ &\geq \frac{n-1}{n+3} \end{aligned}$$

Thus for all n, we must have $c \geq \frac{n-1}{n+3} = 1 - \frac{4}{n+3}$, and therefore c must be at least 1.

9. Prove that $f(m+n+1) = f(m)f(n) + f(m+1)f(n+1)$ for all $n, m \geq 0$, where $f(n) = F_n$, the n^{th} Fibonacci number.

Solution: Base case: $n = 0$. Then $f(0) = 0$ and $f(1) = 1$. We have $f(m+1) = 0 + f(m+1)$ is true for all $m \geq 0$.

Suppose true for all $m \geq 0$ and all $0 \leq n < N$. We have $f(m+N+1) = f(m+1+(N-1)+1)$, so we can apply the induction hypothesis. Then

$$\begin{aligned} f(m+N+1) &= f(m+1)f(N-1) + f(m+2)f(N) \\ &= f(m+1)f(N-1) + \left[f(m) + f(m+1) \right] f(N) \\ &= f(m)f(N) + f(m+1)[f(N-1) + f(N)] \\ &= f(m)f(N) + f(m+1)f(N+1) \end{aligned}$$

as required. By induction on n, this holds for all $m, n \geq 0$.