UBC Math Circle 2019 Problem Set 2

Problems will be ordered roughly in increasing difficulty

1. Find the least positive integer n such that $n!$ ends in exactly 2019 zeroes.

Solution: Observe that the number of trailing zeroes in $n!$ is just the number of times a factor of 5 appears in the numbers $1, 2, ..., n$ (with multiplicity). We can write this number as

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{5^k} \right\rfloor .$$

This immediately gives $n < 10000$.

I used the good old guess-and-check method.

1. Plug in $n = 8000$. This is gives 1998, which is almost 2019.
2. Since 1998 is really close to 2019, just enumerate multiples of 5 starting from 8005 until we hit 2019 factors of 5.
3. The answer is 8090.

2. Classify all numbers with exactly 2019 distinct divisors in terms of the number's prime factorization.

Solution: $2019 = 3 \times 673$, and 673 is prime. By the number of divisors formula if $n = \prod_{i=1}^{r} p_i^{a_i}$, then the number of divisors of n is $\prod_{i=1}^{r}(a_i + 1)$. Hence it must be of the form p^{2018}, or $p^2 q^{672}$ for primes p and q.

3. (Putnam 1991) For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i?

Solution: b_i is eventually constant iff b_0 is a perfect square. If b_k is a perfect square, then $d(b_k) = 0$, and $b_{k+1} = b_k$.

If b_k is not a perfect square, then $b_k = m^2 + r$ for $1 \leq r \leq 2m$. Then $d(b_k) = r$, and $b_{k+1} = m^2 + 2r$. We have $m^2 < b_{k+1} < (m+2)^2$ but $b_{k+1} \neq (m+1)^2$ as $2r \neq 2m+1$. Also, $b_{k+1} > b_k$, is not a perfect square, so the sequence is always increasing.

4. For any n, there exist n consecutive positive integers such that none of them are square-free

Solution: Consider the congruences $x \equiv 0 \pmod{2^2}$, $x + 1 \equiv 0 \pmod{3^2}, \ldots, x + n \equiv 0 \pmod{p_n^2}$ Such a system has a solution by the Chinese Remainder's Theorem.

5. (Putnam 1975) A triangular number is a positive integer of the form $\frac{n(n+1)}{2}$. Show that m is a sum of two triangular numbers iff $4m + 1$ is a sum of two squares.

Solution: If $m = \frac{a(a+1)}{2} + \frac{(b(b+1)}{2}$, then $4m + 1 = (a - b)^2 + (a + b + 1)^2$.

Conversely, if $4m + 1 = a^2 + b^2$, then $a^2 + b^2 \equiv 1 (\text{mod } 4)$. Thus, WLOG, a is even and b is odd. Then $(a + b - 1)$ and $(a - b - 1)$ are both even. Let $c = \frac{(a+b-1)}{2}$ and $d = \frac{(a-b-1)}{2}$. Then $\frac{c(c+1)}{2} + \frac{d(d+1)}{2} = m$.

6. Show that there are infinitely many primes of the form $3n + 1$, without using Dirichlet's prime theorem.

Solution: We consider the polynomial $P(x) = x^2 + x + 1$. We claim that if $p \mid P(x)$, then $p \equiv 1 \mod 3$. We know that $x^3 - 1 = (x - 1)(x^2 + x + 1)$. So, if $p \mid P(x)$, then $p \mid x^3 - 1$. It follows that the order of x modulo p is divisible by 3. But if p is prime, then the order of x must divide $\phi(x) = p - 1$. So, $3 \mid p - 1 \iff p \equiv 1 \mod 3$.

Now suppose there are finitely many primes of the form $3n + 1$. Let these be $p_1, p_2, \ldots, p_k$. Take $Q := \prod_{j=1}^{k} p_j$. Clearly $P(Q) > 1$. So, it must be divisible by some prime, p. We showed above that p must be of the form $3n + 1$. But, p cannot be one of p_j, as then $p \mid Q^2 + Q$, giving $p \mid 1$. The former is impossible, so we must have a new prime of the form $3n + 1$. This contradicts our assumption that there are only k primes of the form $3n + 1$. Thus, there are infinitely many such primes.

7. Show that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{m+1}{n} + \frac{n+1}{m} \in \mathbb{N}$$

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Solution: Some Vieta jumping. https://math.stackexchange.com/questions/2940631/show-that-infinitely-many-positive-integer-pairs-m-n-exist-s-t-fracm1n

8. Let $n \geq 2$ be an integer. Show that $m = 2n^2 - 1$ is the least natural number such that there exist n positive integers $a_1, a_2, \ldots, a_n$ satisfying

   1. $a_1 < a_2 < \ldots < a_n = m$
   2. All of $\frac{a_1^2+a_2^2}{2}, \frac{a_2^2+a_3^2}{2}, \ldots, \frac{a_{n-1}^2+a_n^2}{2}$ are perfect squares.

Solution: Notice that for any positive integers $b > a$, if $\frac{a^2+b^2}{2}$ is a perfect square then b must be somewhat greater than a. We shall make this precise.
Let $b = a + d$ where d is even (if not $\frac{a^2+b^2}{2}$ would not be an integer) then $\frac{b^2+a^2}{2} = \frac{a^2+(a+d)^2}{2} = a^2 + ad + \frac{d^2}{2}$.
Now $a^2 + ad + \frac{d^2}{2} > a^2 + ad + \frac{d^2}{4} = (a + \frac{d}{2})^2$, so in order for $a^2 + ad + \frac{d^2}{2}$ to be a perfect square it must be that $a^2 + ad + \frac{d^2}{2} \geq (a + \frac{d}{2} + 1)^2$. We would like to bound the difference d by a, hence solving this equation gives $d \geq \left\lfloor 2\sqrt{2a+2} \right\rfloor + 2$
Going back to the problem, essentially we want to see how small a_n can be. A natural way to do this is to choose $a_1 = 1$, and try to lower bound $a_2, \ldots a_n$ using the given bound. Indeed, assume that $a_i \geq 2i^2 - 1$ (this is true for a_1) then by what was proven above, $a_{i+1} \geq \left\lfloor 2\sqrt{2a_i+2} \right\rfloor + a_i + 2 = 2(i+1)^2 - 1$. Hence, by induction we have $m = a_n \geq 2n^2 - 1$, which is exactly what we want to prove. For such a value of m we can choose $a_i = 2i^2 - 1$ and the two conditions are satisfied.