UBC Math Circle 2019 Problem Set 3

Problems will be ordered roughly in increasing difficulty

1. Three polygons of area 3 each are placed inside a square of area 6. Prove that there exist 2 polygons with common area at least 1.

Solution: The total area of polygons is 9, in a square of area 6, so there is an overlapping area of at least 3. There are 3 pairs of possible polygons, so at least one pair must have common area at least 1, by pigeonhole.

2. Six distinct positive integers are randomly chosen between 1 and 2019, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

Solution: For the difference to be a multiple of 5, the two integers must have the same remainder when divided by 5. Since there are 5 possible remainders (0-4), by the pigeonhole principle, at least two of the integers must share the same remainder. Thus, the answer is 1. (Art of Problem Solving)

3. Ten points lie in a square with side length 3. Prove that the shortest distance between any pair of points is at most $\sqrt{2}$.

Solution: Divide the square into 9 1×1 squares. Two points must lie in the same square.

4. Among 7 real numbers there exist x, y such that $0 < \frac{x-y}{1+xy} < \frac{1}{\sqrt{3}}$

Solution: For each real number a we may write $a = \tan \alpha$ for some $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then there exist $x = \tan \alpha$, $y = \tan \beta$ among 7 given numbers such that $|\alpha - \beta| < \frac{\pi}{6}$ and $\alpha < \beta$. Then one may check that x, y satisfy the required condition.

5. A lattice point in 2-dimensions is a point (x, y) where both x and y are integers. We colour every lattice point in 2-dimensions one of 2019 colours. Show that there exists an axis-aligned monochromatic rectangle. An axis-aligned rectangle is one whose sides are parallel to the x and y axes. A monochromatic rectangle is a rectangle whose vertices have the same colour.

   Bonus: Show that this statement is true in n-dimensions when rectangle is replaced with an n-dimensional box.

Solution: Consider the points of the for $(x, 1), ..., (x, 2020)$. Out of these 2020 points, we must have two points that are the same colour by pigeonhole. Observe that there are finitely many ways to assign 2019 colours to the 2020 points $(x, 1), ..., (x, 2020)$. This means that if we consider all integers x between 1 and N for some sufficiently large N, we will find two x_1 and x_2 such that (x_1, y) and (x_2, y) are the same colour for all $y = 1, ..., 2020$. This gives an axis aligned monochromatic rectangle.

6. Prove that among any 13 real numbers, there exists two real numbers x and y such that

$$|x - y| \leq \left(2 - \sqrt{3}\right) |1 + xy|.$$

Solution: Same as question 4.

7. Henry picks up pennies for 4 Henry-years (1425 days because a Henry-year has 356 days and there is a leap year every four years). Every day he picks up at least one penny, but he did not pick up more than 2019 pennies in total. Prove that there exists some consecutive days during which Henry picks up exactly 830 pennies.

Solution: Let a_i be the number of pennies Henry picked up starting from the first day to the ith day. Let $b_i = a_i + 830$. If there exists some i and j such that $a_i = b_j$, then between the jth day and the ith day, Henry would have picked up 830 pennies. Note the following:

$$1 \leq a_1 < a_2 < a_3 < ... < a_{1425} < 2019$$

$$831 \leq b_1 < b_2 < b_3 < ... < b_{1425} < 2849$$

Note that we have a total of 2850 numbers, so there are two equal numbers. Because Henry picks up at least one penny a day, the a_is are strictly monotonic, as are the b_is, so there must exist one a_i that's equal to some b_j.

8. (IMO 1983) The points on the perimeter of a equilateral triangle are coloured red or blue. Show that there exists a monochromatic right-angled triangle whose vertices are on the perimeter of the equilateral triangle.

Solution: Partition each edge into three equal parts using two points. These six points form a regular hexagon. Now, we proceed by contradiction.

First, we show that the opposite points on this regular hexagon must have different colours. If the opposite points have the same colour, then no other points on the regular hexagon can have that colour. However, there are three pairs of opposite points, so opposite points must all be of different colours.

Since opposite points are all different colours, there must be two adjacent points that are of different colours. Observe that the points opposite of this pair must also be of different colours. One of these two pairs must lie on the same side of the equilateral triangle. Now, we can show that no points on this side can be of any of these two colours (or we form a monochromatic right-angled triangle), but there are only two colours so we are done.