UBC Math Circle 2019 Problem Set 5

Problems will be ordered roughly in increasing difficulty

1. For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

   Solution: Yes. Label the angles as P_1, Q_1, R_1 for the angles of T_1, and P_2, Q_2, R_2 for the corresponding angles in T_2. As the angles of T_1 and T_2 sum to π, we must have some angle in T_1 that is no more than its corresponding angle in T_2. WLOG, let this be $P_1 \le P_2 < \frac{\pi}{2}$, as T_2 is an acute triangle. Then $\sin(P_1) \le \sin(P_2)$, and we have that $A_1 = \frac{1}{2}x_1y_1\sin(P_1) \le \frac{1}{2}x_2y_2\sin(P_2) = A_2$, where x_i, y_i are the sides adjacent to P_i.

2. Let T be an equilateral triangle, and let P be any point inside T. Let $a = AP$, $b = BP$, and $c = CP$. Show that there exists a triangle with side lengths a, b, and c.

   Solution: Rotate the equilateral triangle $60°$ about one of the vertices. We have the following picture:

PS5Q2.png

3. Suppose that S is a finite set of points in the plane such that the area of triangle ABC is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

Solution: Since S is finite, we can choose three points A, B, C in S so as to maximize the area of the triangle ABC. Let A', B', C' be the points in the plane such that A, B, C are the midpoints of the segments $B'C', C'A', A'B'$; the triangle $A'B'C'$ is similar to ABC with sides twice as long, so its area is 4 times that of ABC and hence no greater than 4.

We claim that this triangle has the desired effect; that is, every point P of S is contained within the triangle $A'B'C'$. (To be precise, the problem statement requires a triangle of area exactly 4, which need not be the case for $A'B'C'$, but this is trivially resolved by scaling up by a homothety.) To see this, note that since the area of the triangle PBC is no more than that of ABC, P must lie in the half-plane bounded by $B'C'$ containing B and C. Similarly, P must lie in the half-plane bounded by $C'A'$ containing C and A, and the half-plane bounded by $A'B'$ containing A and B. These three half-planes intersect precisely in the region bounded by the triangle $A'B'C'$, proving the claim.

4. Given two circles C and C' such that the radius of C is less than the radius of C', and furthermore, C is inside C'. We call C and C' the bounding circles. A "Steiner chain" is a chain of circles inscribed between C and C' such that all the circles in the chain are tangent to both C and C', and furthermore, consecutive circles in the chain are tangent. This is best illustrated via a picture.

   `steinerchain.png`

   Let C and C' be given bounding circles. Suppose that there exists some Steiner chain. Prove that there are infinitely many Steiner chains, and furthermore, the tangent points between circles in the chain (so not the tangent points with C and C') all lie on some third circle.

   Solution: Use circle inversion on the whole thing, so that C and C' become concentric. Then we can rotate the Steiner chain about the common centre, and uninvert, so we get infinitely many Steiner chains. Furthermore, in the inverted picture, the points of tangency lie on some circle C'' which lies between C and C'. When we uninvert C'', we get that the tangent points of the original Steiner chain lie on a circle.

5. Given an acute angled triangle ABC, let the midpoints of the sides BC, CA and AB be D, E and F, respectively. Let the foot of the altitude of the triangle starting from C be T_1. Let l be some line passing through point C, but not containing T_1, and let T_2 and T_3 be the feet of the pependiculars from l to A and B, respectively. Prove that the circle DEF passes through the center of the circle $T_1T_2T_3$.

Solution:

PS5picture.png

Let J be the centre of the cirlcle $T_1T_2T_3$, and let $G, H, I \equiv T_1$ the feet of the altitudes of ABC. Since the circumcircle of the median triangle DEF is the nine-point-circle of ABC, D, E, F, G, H, I are concyclic. Since $BT_3C = CIB = \frac{\pi}{2}$, BCT_3T_1 is a cyclic quadrilateral and the perpendicular bisector of T_1T_3 goes through D. Let T_4 be the midpoint of T_2T_3. By Thales' theorem, the perpendicular bisector of T_2T_3 goes through F. Now the angle between the FT_4 and the JD lines equals the angle between the CT_3 and T_1T_3 lines, by switching to perpendiculars. By exploiting the ciclicity of BCT_3T_1 we have that this angle equals $CBA = FED$. If J is the intersection between FT_4 and the perpendicular bisector of T_1T_3, J is the circumcenter of $T_1T_2T_3$ and $FJD = FED$. It follows that J belongs to the nine-point-circle of ABC, as wanted.

6. Give a tiling of $\mathbb{R}^3$ by circles of finite, positive radius. Here by tiling we mean give a set of circles such that for each point in $\mathbb{R}^3$, it lies on exactly one of the circles.

Solution: Consider circles of the form $(x - 4k - 1)^2 + y^2 = 1$, $z = 0$. These are circles of radius 1 lying in the $x - y$ plane. Now for every $r > 0$, consider the sphere $S = x^2 + y^2 + z^2 = r^2$. We have that S intersects the previous set of circles at exactly 2 distinct points. Thus if we show that every sphere with 2 points removed can be covered with circles, then we are done.

Case 1: the two removed points are antipodal. WLOG assume that they are the north pole and the south pole. Then the rest of the sphere can be covered with circles of latitude.

Case 2: the two removed points are not antipodal, so the tangent planes to the sphere at the two points are not parallel. Thus they intersect in a line, L. For every point on the sphere, construct a plane passing through the point and L. The intersection of this plane with the sphere is a circle, and every point on this circle generates the same plane, so each point on the sphere lies on exactly one such circle. This fails at the two removed points, as there we have degenerate circles of radius 0, but we're removing them anyways.

Thus every sphere with any two distinct points removed can be covered with circles.

7. Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to Γ, and let t_a, t_b and t_c be the lines obtained by reflecting t across the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Solution: To avoid a large case distinction, we will use the notion of oriented angles. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo $180°$.

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Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, and $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter. Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'' , respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(T, TC'') + \angle(TC''B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'') \\ &= 2(\angle(t, TC) + \angle(TC, BC)) = 2\angle(t, BC) = \angle(t, t_a)\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b || A''C''$ and $t_c || A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired. We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.
Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. $\square$

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.
Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$. Due to symmetry, the line DB is one of the angle bisectors of the lines BD and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90°$, so

$$\angle(B'B, BC) = \angle(B'B, B'D) = 90° - \angle(BC, DF) - \angle(DF, BA) = 90° - \angle(BC, AB)$$

Analogously, we get $\angle(C'C, B'C') = 90° = \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC)$$

which means exactly that the points A, B, I, C are concyclic. $\square$

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying Pascal's theorem to the hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the inersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$, which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.