UBC Math Circle 2019 Problem Set 6

Problems will be ordered roughly in increasing difficulty

1. Prove the AM-GM inequality using Jensen's Inequality. As a reminder AM-GM is as follows: Given non-negative real numbers $a_1, a_2, ..., a_n$, we have

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

with equality if and only if $a_1 = a_2 = ... = a_n$. Jensens inequality states given a convex function F and real numbers $x_1, x_2, ..., x_n$, as well as non-negative real numbers $a_1, a_2, ..., a_n$, then we have

$$(a_1 + a_2 + ... + a_n)F\left(\frac{a_1 x_1 + a_2 x_2 + ... + a_n x_n}{a_1 + a_2 + ... + a_n}\right) \leq a_1 F(x_1) + a_2 F(x_2) + ... + a_n F(x_n)$$

Bonus question: Prove AM-GM using (cauchy) induction.

Solution: Use $F(x) = -\log(x)$, which is convex. Let $a_i = \frac{1}{n}$ for all i, and apply Jensen's inequality to get

$$-\log\left(\frac{x_1 + ... + x_n}{n}\right) \leq -\frac{1}{n}\left[\log(x_1) + ... + \log(x_n)\right]$$

Then we have

$$\log\left(\frac{x_1 + ... + x_n}{n}\right) \geq \log\left((x_1 ... x_n)^{\frac{1}{n}}\right)$$

Now we apply the fact that $\log(x)$ is an increasing function to conclude that

$$\frac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 ... x_n}$$

The bonus problem is a standard problem in cauchy induction, and finding it via google is left as an exercise.

2. Let x, y, z be positive real numbers. Show that

$$x^3 y + y^3 z + z^3 x \geq x^2 yz + xy^2 z + xyz^2.$$

Solution: Divide both sides by xyz to get

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \geq x + y + z. = \frac{x^2}{x} + \frac{y^2}{y} + \frac{z^2}{z}.$$

Without loss of generality, let $x \leq y \leq z$. Then $x^2 \leq y^2 \leq z^2$ and $\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$, so the above inequality follows from the rearrangement inequality.

3. Show for all positive reals a, b, c, d:

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$$

Solution: Let $F(x) = \frac{1}{x}$, which is convex on the positive reals. Then,

$$\frac{1}{8}F(8a) + \frac{1}{8}F(8b) + \frac{1}{4}F(4c) + \frac{1}{2}(2d) \geq F(a+b+c+d)$$

by Jensen's. Multiplying by 64 on both sides gives the desired result.

4. Let P be a point inside $\triangle ABC$. Let D, E, F be the feet of the perpendiculars from P to BC, CA, and AB respectively. For which points P is the product $|PD| \cdot |PE| \cdot |PF|$ maximized?

Solution: We write P in barycentric coordinates with respect to $\triangle ABC$, that is $P = (a, b, c)$ and $a + b + c = 1$. Let h_a, h_b, and h_c be the heights from vertices A, B, and C respectively. Observe that $|PD| = ah_a$, $|PE| = bh_b$, and $|PF| = ch_c$, whence the product that we want to maximize becomes $abc(h_a h_b h_c)$. Since $h_a h_b h_c$ is a constant, we actually want to maximize abc when $a + b + c = 1$. We can use AM-GM to conclude that the maximum is achived when $a = b = c = 1/3$. Hence, P must be the centroid of $\triangle ABC$.

5. Let a, b, c, and d be positive real numbers. Show that

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+d)(b+c)}.$$

Hint: Can we introduce a well known function that takes on values between 0 and 1?

Solution: Observe that the inequality is equivalent to

$$\sqrt{\frac{a}{a+d}\frac{b}{b+c}}+\sqrt{\frac{c}{b+c}\frac{d}{a+d}}\leq 1$$

Since $a, b, c, d > 0$, we know that $0 \leq \frac{a}{a+d} \leq 1$, and similarly for the other fractions. Let

$$\sin^2(\alpha)=\frac{a}{a+d},$$
$$\sin^2(\beta)=\frac{b}{b+c}.$$

Then, substituting these trig functions into the equation gives

$$\sin(\alpha)\sin(\beta)+\cos(\beta)\cos(\alpha)=\cos(\alpha-\beta).$$

The last value is a cosine, which is clearly at most 1, so we are done.

6. Let x_i be positive reals such that $\sum_{i=1}^{n}\frac{1}{2019+x_i}=\frac{1}{2019}$. Show that $\frac{\sqrt[n]{x_1x_2\ldots x_n}}{n-1}\geq 2019$

Solution: By AM-GM, for any $1 \leq j \leq n$ we have

$$\frac{x_j}{2019(2019+x_j)}=\sum_{i\neq j}\frac{1}{2019+x_i}\geq (n-1)\frac{1}{\sqrt[n-1]{\prod_{i\neq j}(2019+x_i)}}$$

By multiplying,

$$\prod_{i=1}^{n}\frac{x_i}{2019}\geq (n-1)^n$$

7. For $x, y \in \mathbb{R}^+ : x^2 + y^2 = 1$, find the maximum value of $M = \sqrt{x} + \sqrt{2y}$ using the Cauchy-Schwarz-Bunyakovsky inequality.

Solution:

$$(x^2+y^2)(1+k)\geq (x+\sqrt{k}\,y)^2 \qquad (1)$$
$$(x+\sqrt{k}\,y)(1+k)\geq (\sqrt{x}+k^{3/4}\sqrt{y})^2 \qquad (2)$$

Now multiplying inequalities (1) and $(2)^2$ together gives

$$(x^2+y^2)(1+k)^3\geq (\sqrt{x}+k^{3/4}\sqrt{y})^4$$

Setting $k = 2^{2/3}$ makes the RHS what we want, which is the same as the Holder's inequality above. You can work out that equality in both labeled inequalities requires $x : y = 1 : \sqrt{k}$, which is possible for any $k > 0$.

8. Let a, b, and c be the side-lengths of a triangle. Show that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Hint: One of the above problems might help (and not the other triangle one).

Solution: We can substitute $a = y+z$, $b = z+x$, and $c = x+y$. Note that $x = s-a$, $y = s-b$, and $z = s-c$, and s is the semi-perimeter. Now, the inequality becomes (after a lot of work and a lot of abusing symmetry)

$$x^3y + y^3z + z^3x \geq x^2yz + xy^2z + xyz^2,$$

which is true by the rearrangement inequality.

9. Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d = 4$.

Solution: We observe

$$\frac{1}{b^3+4} \geq \frac{1}{4} - \frac{b}{12}$$

since $12 - (3-b)(b^3+4) = b(b+1)(b-2)^2 \geq 0$. Moreover,

$$ab + bc + cd + da = (a+c)(b+d) \leq \left(\frac{a+b+c+d}{2}\right)^2 = 4.$$

Thus

$$\sum_{\text{cyc}} \frac{a}{b^3+4} \geq \frac{a+b+c+d}{4} - \frac{ab+bc+cd+da}{12} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

This minimum $\frac{2}{3}$ is achieved at $(a, b, c, d) = (2, 2, 0, 0)$ and permutations.