UBC Math Circle 2019 Problem Set 7

Problems will be ordered roughly in increasing difficulty

1. Consider all 2019-element subsets of the set $\{1, 2, ..., 10000\}$. From each subset choose the least element. Find the arithmetic mean of all these least elements.

Solution: Each 2019 element subset $\{a_1, a_2, ..., a_{2019}\}$ of $\{1, 2, ..., 10000\}$ with $a_1 < a_2 < ... < a_{2019}$ contributes a_1. Now consider the set $\{a_1 + 1, a_2 + 1, ..., a_{2019} + 1\}$. There are a_1 ways to choose a positive integer k such that $k < a_1 + 1 < ... < a_{2019} + 1$. Thus, the number of ways to choose the set $\{k, a_1 + 1, a_2 + 1, ..., a_{2019} + 1\}$ must be equal to the least element sum. But, this is the number of ways to choose a 2020 element subset from $\{1, 2, ..., 10001\}$.

So, the average is given by:

$$\frac{\binom{10001}{2020}}{\binom{10000}{2019}} = \frac{10001}{2020}$$

Source: 2015 AIME I Problem 12 by Art of Problem Solving

2. There are 2019 students at an university. Students join together to form several clubs (a student may belong to many different clubs). Some clubs join together to form societies (a club may belong to many different societies. There are a total of k societies. Suppose that the following conditions hold:

   1. Each pair of students are in exactly one club together.
   2. For each student and each society, the student is in exactly one club of the society.
   3. Each club has an odd number of students. In addition, a club with $2m+1$ students is in exacty m societies.

   Find all possible values of k.

Solution: Replacing the number 2019 with the variable n, we will count the number of ordered triples (a, C, S), where a is a student belonging to a club C, which belongs to a society S. We will denote such triples acceptable.

Now, for any student a and any society S, there is exactly one club which will form an acceptable triple. Thus the number of triples is nk.

Consider any club C with $|C|$ members. It is in $\frac{|C|-1}{2}$ societies, so C can form $\frac{|C|(|C|-1)}{2}$ acceptable triples. If $\mathcal{C}$ denotes the set of all clubs, then this implies that $nk = \sum_{C \in \mathcal{C}} \frac{|C|(|C|-1)}{2} = \sum_{C \in \mathcal{C}} \binom{|C|}{2}$.

But since any pair of students belong to exactly one club, it follows that $\binom{n}{2} = \sum_{C \in \mathcal{C}} \binom{|C|}{2}$, or $\frac{n(n-1)}{2} = nk$. Therefore $k = \frac{n-1}{2}$. Thus, if there are 2019 students, there have to be exactly 1009 societies

Source: 2004 IMO Shortlist Problems/C1 by Art of Problem Solving

3. At the vertices of a regular hexagon are written six non-negative integers whose sum is 2019. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Solution: Assume the original numbers are a, b, c, d, e, f. Since $a+b+c+d+e+f$ is odd, either $a+c+e$ or $b+d+f$ must be odd. WLOG let $a+c+e$ be odd and $a \geq c \geq e \geq 0$.

Case 1 $a, c, e > 0$. Define Operation A as the sequence of moves from Step 1 to Step 3, shown below:

Notice that Operation A changes the numbers a, c, e to $c-e, c, a-c$ and they are all nonnegative, since $a \geq c \geq e$. Their sum changes from $a+c+e$ to $a+c-e$; it decreases as long as $e \neq 0$. If we repeat Operation A enough times, its sum will decrease and eventually we will arrive at a point where at least one of the numbers in the positions originally occupied by a, c, e has become a 0.

Case 2 $a, c > 0$ and $e = 0$. Define Operation B as the sequence of moves from Step 1 to Step 3, shown below:

where in Step 3, we take the nonnegative choice of $2c-a$ or $a-2c$. $a, c, 0$ is changed to either $c, 2c-a, 0$ or $c, a-2c, 0$. If we have $c, 2c-a, 0$, their sum is $3c-a$ and this is less than $a+c+0$ (the original sum) unless $a = c$, but $a \neq c$ since the original sum $a+c+0$ is odd by assumption. If we have $c, a-2c, 0$, their sum is $a-c$, which is less than $a+c$. Operation B applied repeatedly will cause either a or c to become 0.

Case 3 $a > 0$ and $c = e = 0$. Define Operation C as the sequence of moves from Step 1 to Step 4, shown below:

Source: 2003 USAMO Problems/Problem 6 by Art of Problem Solving

4. How many paths on the surface of an $n \times n \times n$ cube travel from $(0,0,0)$ to (n,n,n) while taking only unit steps in the positive x, y, or z direction?

Solution: Each path traverses two faces, and there are 6 possible pairs of faces to traverse. Each of these two face paths is a $2n \times n$ rectangle, so we get a total of $6\binom{2n+n}{n}$ paths. However, we have overcounted paths which fully traverse edges. Consider paths which fully traverse at least one edge. There are 6 possible edges for these paths to include, and there are $\binom{n+n}{n}$ ways to traverse the remaining face. Thus, we now have $6\binom{3n}{n} - 6\binom{2n}{n}$ paths so far. There is no way for a path to traverse exactly two edges, so we just need to see how many paths that traverse three edges we over counted by. Such paths are counted three times each in the original count, but also twice each when considering paths that traverse at least one edge, so the final count is correct. Thus we have $6\left[\binom{3n}{n} - \binom{2n}{n}\right]$

5. Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \leq x, y, z \leq 9$. Players A and B play the following game. Player A chooses a triple (x, y, z) in T, and Player B has to discover A's triple in as few moves as possible. A move consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number $|x+y-a-b| + |y+z-b-c| + |z+x-c-a|$. find the minimum number of moves that B needs to be sure of determining A's triple (IMO shortlist 2002)

Solution: In mod 2, we see that

$$|x+y-a-b| + |y+z-b-c| + |z+x-c-a| \equiv 2(x+y+z-a-b-c),$$

so the outcome of B's move must always be even. Furthermore, the outcome must be no greater than 54 and no less than 0, so there are at most 28 different possible outcomes per move. Since there are 10^3 possible triples (x, y, z) and at most $28^2 < 10^3$ possible outcomes after two moves, at least three moves are required.

We will now show how to determine (x, y, z) in three moves. A first move of $(0, 0, 0)$ will give us $2(x+y+z)$. We shall denote $x+y+z$ as s.

If $s \leq 9$, then the moves $(9, 0, 0)$ and $(0, 9, 0)$ will give us $18 - 2x$ and $18 - 2y$, respectively, enabling us to determine (x, y, z). Similarly, if $s \geq 18$, the moves $(0, 9, 9)$ and $(9, 0, 9)$ will give us $2x$ and $2y$.

If $9 < s < 18$, then our second move is $(9, s-9, 0)$. Let us call the result $2k$. We have two cases. In Case I, $y > s-9$, which gives us $x = 9-k$, $z < k$. In Case II, $y \leq s-9$, so $x \geq 9-k$ and $z = k$. In either case, we have $z \leq k \leq y+z$ (the right-hand side comes from $y+z = s-9+k$ or $z = k, y \geq 0$) and $x+z \leq 9+k$.

Now, if $s-k \leq 9$, then our third move is $(s-k, 0, k)$. This gives us

$$|x+y-s+k| + |y+z-k| + |z+x-s| = k-z+y+z-k+y = 2y,$$

which gives us y and tells us whether Case I or Case II holds, letting us determine (x, y, z).

On the other hand, if $s-k > 9$, our third move is $(9, s-k-9, k)$. This gives us

$|x+y-s+k|+|y+z-s+9|+|z+x-k-9| = |k-z|+|9-x|+|k+9-z-x| = 2(k+9-s+y)$,

which again gives us y, telling us which of Cases I and II hold, letting us determine the triple (x, y, z)

6. A partition of a positive integer n is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are said to be the same partition. Show that the number of ways to partition n with only odd summands (e.g. $4 = 1+1+1+1 = 1+3$) is equal to the number of ways to partition n with only distinct summands (e.g. $4 = 1+3 = 4$).

Solution: We admit the following result: There is a function known as a *generating function*, $f_k(x) = 1 + x^k + x^{2k} + ...$, which gives the number of ways of partitioning n into summands, all of which are exactly k, in the sense that the number of such partitions is the coefficient of the x^n term. Similarly, to get the the number of ways of partitioning n into summands, all of which are at most k, has generating function $f_1(x)f_2(x)...f_k(x)$, and the number of partitions is once again the coefficient of the x^n term. Thus, to show the number of partitions with odd summands is equal to the number of partitions with distinct summands, it sufficies to show the generating functions are the same.

We ignore issues of convergence (for example we assume $|x| < 1$ to use the geometric series identities). The generating function for the number of partitions with only odd summands is

$$g(x) = f_1(x)f_3(x)f_5(x)\cdots = \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$$

The generating function for partitions into distinct parts is

$$\begin{aligned} h(x) &= (1+x)(1+x^2)(1+x^3)\cdots = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= \frac{\prod_{i=1}^{\infty}(1-x^{2i})}{\left(\prod_{i=1}^{\infty}1-x^{2i-1}\right)\left(\prod_{i=1}^{\infty}1-x^{2i}\right)} = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}} \end{aligned}$$

7. Find all naturals k such that given k distinct, pairwise non-parallel lines on the plane, we can write a number in $\{1, 2, \ldots k-1\}$ on each intersection so that every line has all numbers from 1 to $k-1$.

Solution: Graph theoretic solution:
Consider each line to be a vertex and each intersection of pairs of lines to be an edge. This forms a complete graph. Labels on intersections are now labels on edges. Clearly no two edges labeled with the same number can be incident to the same vertex.

This problem is equivalent to decomposing the complete graph on k vertices into $k-1$ prefect matchings, where we'll label the edges of each matching with a unique number. A result in graph theory shows that this is possible whenever k is even. Clearly this is impossible when k is odd.