UBC Math Circle 2019 Problem Set 8

Problems will be ordered roughly in increasing difficulty

1. Consider a game of chess where a player makes two consecutive moves before it is their opponent's turn, and there is Show that the second player has no winning strategy.

> **Solution:** Suppose the second player, B, has a winning strategy. The first player, A, can move one of their knights out, and then back in. Thus the board has reset to its original state, except now B goes first, and A second. Thus A can copy B's winning strategy, meaning B can't win.
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> A slight concern is that the board is not completely returned to its original state, as moves have been made. This affects rules such as the trifold repetition and fifty move rule. However, the only consequence of these rules is that B's winning strategy might become a drawing strategy when A employs it, which is fine because B still doesn't win.

2. Alphonse and Beatrice play a very interesting game "Put a Knight!" on a chessboard $n \times n$ in size. In this game they take turns to put knights (a type of chess piece) on the board so that no two knights could threaten each other. The player who can't put a new knight during his move loses. For each n, determine which player wins considering that both players play optimally well and Alphonse starts.

> **Solution:** The player with the winning strategy wins by placing a knight on the square opposite to where the opponent placed the last knight (check that this is valid). If n even, Beatrice wins. If n odd, Alphonse wins, by placing the first knight on the middle square.

3. There are 2019 jellybeans on a table. Paul and Henry play a game removing at most 31 jellybeans and at least 1 jellybean. Paul goes first.

   (a) If the player who takes the last jellybean wins, who would win if they both play optimally?

   (b) What if there are n jellybeans and each player removes at least k jellybeans. Who would win?

   (c) What if the person who takes the last jelly bean loses, then who would win?

Solution:

(a) Paul would win.

(b) Paul will win as long as n is not divisible by k. Otherwise Henry wins. The winning strategy is to always keep the total number of jellybeans at $0 \pmod{k}$.

(c) The winning strategy is to keep the sum at $1 \pmod{k}$.

4. Paul and Henry are eating a basket of 3584 blueberries. They've decided that whoever eats the last blueberry will win. The loser will pay for the basket of blueberries, so neither of them want to lose. Henry goes first. To be fair, if there are n blueberries left, they can eat either 1 blueberry or $\lceil \frac{n}{2} \rceil$ blueberries. If they both play optimally, who will pay for the basket of blueberries?

Solution: Henry will lose.

Let's call numbers where the first player wins winning positions, and the rest of the positions losing positions. You can prove by induction that all odd numbers are winning positions.

To determine if an even number n is a winning state, let 2^k be the largest power of 2 that divides n. If k is even, the position is winning, otherwise it is losing. This can also be proved via induction.

Since, $3584 = 7 \cdot 2^9$ then Henry loses.

5. We have a row of spoons and forks. In their turn, a player has to put one or more pieces of cutlery away, but they all have to be of the same kind, and they all have to be consecutive items from the right end of the row. For instance, if the row is SFFSSSFFFF, then the player may put one, two, three, or four forks away. If the row is FSFFSSSFS then the player has to put away the right-most spoon. The winner is the player who puts away the last item. Give a winning strategy for one of the players.

Solution: A run of length k consists of k consecutive letters that are all the same. We have two cases:

1. There are no runs of consecutive letters longer than 1. Then there is never any choice, and the first player wins if the number of letters is odd, otherwise, the second player wins.

2. There is at least one run of consectutive letters longer than 1. Then the winning player is the first player, whom we shall call A, to encounter such a run. This is by induction: If this run is the left-most run, then it is clear. If it is not, then when A reaches it, they have two options: either remove the entire run, or all but one letter of the run. By induction, one player must have a winning strategy for the letters remaining after the whole run is removed. If it is the first player to move, then A leaves a single letter, which B has to remove. Then it is A's turn to move, and they apply the winning strategy. Otherwise, the second player to move from that state has the winning strategy, in which case A removes the entire run, leaving B to be the first player in a situation where the second player wins.

6. Princess Acute has two suitors: Prince Scalene and Prince Isosceles. They alternate days in trying to gain her heart by bringing her roses. They pick up the roses from the same garden. Each day, the corresponding suitor will bring her one or two roses, but never two roses of the same colour. The suitor to pick the last rose from the garden will win Acutess heart. Assume perfect courtship, as per usual with these kinds of games.

   1. Assuming that initially there are 3 blue roses, 4 red roses, 5 yellow roses, and 6 white roses, who will win?
   2. If there are m blue roses, n red roses, and k yellow roses, who will win?

   The general case for ≥ 6 different colours is an open problem!

Solution:

1. The first suitor, Prince Scalene, will win. The strategy starts with taking one blue rose and one yellow rose, leaving an even number of roses in each colours. Then no matter what Prince Isosceles does, he cannot win, as there will always be some colour with an odd number of roses. Prince Scalene then just mirrors Prince Isosceles's move, again leaving an even number of roses in each colour.
2. To solve this problem, we need to introduce some game theory notation. A winning state is one in which the next player to make a move will win. A losing state is one in which the next player to make a move will lose.

   We prove that if there is an even number of every colour rose, this is a losing state. This is by induction. If every pile has 0 roses in it, then the next player, call them A, has already lost. If not, then A leaves an odd number of roses in every piles they take roses from. Then their opponent, B, can mirror their move to get back to a state with all even numbers, which we know that A loses by induction.

Now, if we consider a state with 2 even numbers and 1 odd number, we have that the next player, A, can remove a single rose from the pile with an odd number, leaving an even number of roses, which is a losing state for B. Thus 2 even numbers and 1 odd number of roses is a winning state.

Similarly, if we start with 1 even number and 2 odd numbers, then A can remove a single rose from each of the odd piles, leaving a losing state for B. Thus this is a winning state.

Finally, if we have 3 odd numbers, this is a losing state, as no matter what move A makes, they go to one of the previous winning states for B.

Thus, in our problem, Prince Isosceles will win if the parities of all the piles of roses are the same, and Prince Scalene will win otherwise.