UBC Math Circle 2019 Problem Set 9

Problems will be ordered roughly in increasing difficulty

1. The equation $x^4 + Ax^3 + Bx^2 + Cx + D = 0$ has solutions $x = 3 \pm \sqrt{2}, 1 \pm \sqrt{5}$. Find $A + B + C + D$.

> **Solution:** Let $P(x) = x^4 + Ax^3 + Bx^2 + Cx + D$. We have $P(1) = 1 + A + B + C + D$. Since P is monic, we also know $P(x) = (x-3-\sqrt{2})(x-3+\sqrt{2})(x-1-\sqrt{5})(x-1+\sqrt{5})$ so plug in 1 to get $A + B + C + D + 1 = -10 \Rightarrow A + B + C + D = \boxed{-11}$.

2. Find all rational polynomials $p(x) = x^3 + ax^2 + bx + c$ such that a, b, c are roots of p.

> **Solution:** By Vietas formula
>
> $$\begin{cases} a + b + c & = -a \\ ab + bc + ac & = b \\ abc & = -c \end{cases}$$
>
> From the third equation $(ab + 1)c = 0$. Thus $ab = -1$ or $c = 0$. If $c = 0$, then $a + b = -a$ and $ab = b$. Hence $(a, b, c) = (0, 0, 0)$ or $(1, -2, 0)$.
>
> If $ab = -1$, then $c = -2a - b$ and
>
> $$\begin{aligned} -1 + b(-2a - b) + (-2a - b)a &= b \\ 2a^2 - 2 + b + b^2 &= 0 \\ 2a^4 - 2a^2 + a^2 b + a^2 b^2 &= 0 \\ 2a^4 - 2a^2 - a + 1 &= 0 \end{aligned}$$
>
> Since a is rational, the only solution is $a = 1$ and $(a, b, c) = (1, -1, -1)$.

3. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where a, b, c are three different integers. Prove that $p(x)$ has no integral zeros.

> **Solution:** Suppose for contradiction that p has an integral zero. Then we can write $p(x) = (x - d)Q(x)$ for some integer d. Then we get
>
> $$\begin{aligned} -1 &= (a - d)Q(a) \\ &= (b - d)Q(b) \\ &= (c - d)Q(c) \end{aligned}$$

Note that $Q(k) \in$ if $k \in$, and similarly, $k - n \in$ for $k, n \in$. Now the only way to multiply two integers to get -1 is for one of them to be $+1$ and the other to be -1. Thus, by pigeonhole, two of $a - d, b - d, c - d$ are the same, which implies that two of a, b, c are the same, a contradiction.

4. A monic polynomial, p, of degree 4 satisfies $p(1) = 10$, $p(2) = 20$, and $p(3) = 30$. Determine $p(12) + p(-8)$.

Solution: We have that $p(x) - 10x$ is monic and has roots at $1, 2, 3$. Thus $p(x) - 10x = (x-1)(x-2)(x-3)(x-c)$. Then $p(12) = 11(10)(9)(12-c) + 120$ and $p(-8) = -9(-10)(-11)(-8-c) - 80$. Thus $p(12) + p(-8) = 990(12-c) + 990(8+c) + 40 = 990(20) + 40 = 19840$.

5. A polynomial P with integer coefficients is called *tri-divisible* if $P(x)$ is a multiple of 3 for all integers x. Determine necessary and sufficient conditions for P to be tri-divisible.

Solution: $P(0), P(1), P(-1) \equiv 0(3)$ are necessary and sufficient (which can easily be seen by reducing the equation mod 3).

6. Determine all polynomials P for which $P(x)^2 - 2 = 2P(2x^2 - 1)$.

Solution: Denote $P(1) = a$. We have $a^2 - 2a - 2 = 0$. Then we have that $P(x) - a$ has a root at 1, so $P(x) - a = (x-1)P_1(x)$, or $P(x) = (x-1)P_1(x) + a$. Substituting this into the initial relation and simplifying gets us $(x-1)\left[(x-1)P_1(x)^2 + 2aP_1(x)\right] = 4(x^2-1)P_1(2x^2-1) - a^2 + a + 2$. Thus for $x \neq 1$, we get $(x-1)P_1(x)^2 + 2aP_1(x) = 4(x+1)P_1(2x^2-1)$. By continuity, we get that the same expression must hold for $x = 1$ as well, as long as it is finite. Evaluating at $x = 1$ gets us $2aP_1(1) = 8P_1(1)$. Now, as $a \neq 4$ (which can be verified by the quadratic formula), we see that either $P_1(1) = 0$, or $P_1(1)$ is infinite. Of course, the latter option isnt possible as P is a polynomial. Thus, we can write $P_1(x) = (x-1)P_2(x)$, giving us $P(x) = (x-1)^2P_2(x) + a$. We can repeat this indefinitely, which is problematic as we should be bounded by the degree of P. More precisely, suppose that $P(x) = (x1)^nQ(x) + a$, where $Q(1) \neq 0$. Once again, substituting into the initial relation and simplifying yields $(x-1)^nQ(x)^2 + 2aQ(x) = 4(x+1)^nQ(2x^2-1)$, giving us $Q(1) = 0$, a contradiction. It follows then that $P(x) = a$, of which the precise value is left as an exercise in the quadratic formula.

7. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q_k(x) = P(P(...P(P(x))...))$, where P occurs k times. Prove that there are at most n integers t such that $Q_k(t) = t$.

Solution: If there is at most one integer t satisfying $Q_k(t) = t$, then we are done. Otherwise, let s, t be integers such that $Q_k(s) = s$, $Q_k(t) = t$. As $P(x)$ is a polynomial with integral coefficients, $u - v|P(u) - P(v)$ for any integers u, v. So

$$s - t|P(s) - P(t)|Q_2(s) - Q_2(t)| \dots |Q_k(s) - Q_k(t) = s - t$$

and hence both $s-t|P(s)-P(t)$ and $P(s)-P(t)|s-t$. This implies that $P(s)-P(t) = s - t$ or $P(s) - P(t) = t - s$. In other words,

$$P(s) - s = P(t) - t \quad \text{or} \quad P(s) + s = P(t) + t \tag{1}$$

It is impossible to have $P(s) - P(t) = s - t$ and $P(u) - P(t) = t - u$ for distinct integral roots s, u, t of the equation $Q_k(x) = x$. Otherwise $P(s)-P(u) = s-t-(t-u) = s + u - 2t$. But $P(s) - P(u) = s - uoru - s$. In either cases, it yields $s = t$ or $u = t$. So only one equation in (1) is true for all the integer roots of $Q_k(x) = x$.

In either case, let us fix t. Then all integral roots of $Q_k(x) = x$ are also, at the same times, roots of the equation $P(x)x = 0$ or $P(x) + x = 0$. Note that $P(x)x$ and $P(x) + x$ are polynomials of degree n, so there are at most n such roots.