UBC Math Circle 2020 Problem Set 1

Problems will be ordered roughly in increasing difficulty

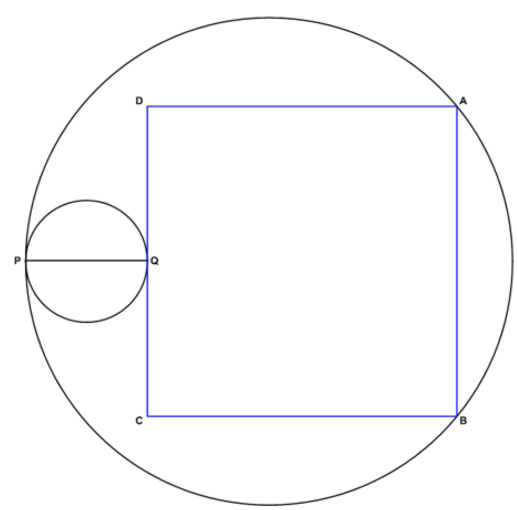
1. (German FMC 2006-2007) Points E and F are taken on the sides AC and BC of ΔABC respectively, such that $AE = BF$. The circles passing through A, C, F and through B, C, E intersect again at the point D. Prove that the line CD bisects $\angle ACB$.

> **Solution:** We want to show $\angle DCA = \angle DCF$. Since $\angle DCF = \angle DAF$ and $\angle DCA = \angle DFA$ ($ACFD$ is an inscribed equilateral), we would like to show that $\angle DAF = \angle DFA$, which can be achieved by showing $AD = DF$ (alternatively you can also show $BD = DE$).
>
> Now, since $ACFD$ is an inscribed equilateral, we have $\angle DAE + \angle DFC = \pi$, so $\angle BFD = \angle DAE$. Similarly, since $BDEC$ is also an inscribed equilateral, we have $\angle AED = \angle DBF$. With the hypothesis that $AE = BF$, we conclude that the triangles AED and FBD are congruent (angle-side-angle). Hence $AD = DF$, and we are done.

2. (PRMO 2012) $ABCD$ is a square with $AB = 1$. Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY?

> **Solution:** Let E be the intersection between AY and DX. We have that the ratio of XY to AD is the same as EY to AE. We also have that $1 = AY = AE + EY$, and $AE = \frac{\sin(30)}{\sin(120)} = \frac{1}{\sqrt{3}}$. Then $XY = \frac{1-AE}{AE} = \sqrt{3} - 1$.

3. (AIME 1994) A circle with diameter $\overline{PQ}$ of length 10 is internally tangent at P to a circle of radius 20. Square $ABCD$ is constructed with A and B tangent on the larger circle, $\overline{CD}$ tangent at Q to the smaller circle and the smaller circle outside $ABCD$. Find the length of $\overline{AB}$ written in the form $m + \sqrt{n}$.



Solution: Call the center of the larger circle O. Extend the diameter $\overline{PQ}$ to the other side of the square (at point E), and draw $\overline{AO}$. We now have a right triangle, with hypotenuse of length 20. Since $OQ = OP - PQ = 20 - 10 = 10$, we know that $OE = AB - OQ = AB - 10$. The other leg, AE, is just $\frac{1}{2}AB$. Apply the Pythagorean Theorem:

$$\begin{aligned}(AB-10)^2 + \left(\frac{1}{2}AB\right)^2 &= 20^2 \\ AB^2 - 20AB + 100 + \frac{1}{4}AB^2 - 400 &= 0 \\ AB^2 - 16AB - 240 &= 0\end{aligned}$$

Thus the answer is $8 + \sqrt{304}$.

4. (Putnam 2019 A2) In the triangle ΔABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\arctan(\frac{1}{3})$. Find α.

Solution: Let M and D denote the midpoint of AB and the foot of the altitude from C to AB, respectively, and let r be the inradius of ΔABC. Since C,G,M are collinear with $CM = 3GM$, the distance from C to line AB is 3 times the distance from G to AB, and the latter is r since $IG||AB$; hence the altitude CD has length $3r$. By the double angle formula for tangent, $\frac{CD}{DB} = \tan\beta = \frac{3}{4}$, and so $DB = 4r$. Let E be the point where the incircle meets AB; then $EB = \frac{r}{\tan(\frac{\beta}{2})} = 3r$. It follows that $ED = r$, whence the incircle is tangent to the altitude CD. This implies that $D = A$, ABC is a right triangle, and $\alpha = \frac{\pi}{2}$.

5. (USAMO 1996) Let ABC be a triangle. Prove that there is a line l (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in l has area more than $\frac{2}{3}$ the area of triangle ABC.

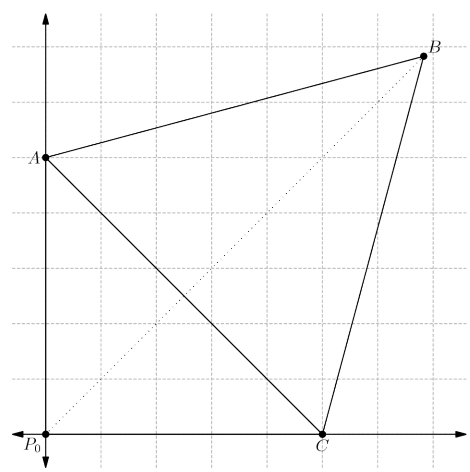
Solution: Let the triangle be ABC. Assume A is the largest angle. Let AD be the altitude. Assume $AB \leq AC$, so that $BD \leq BC/2$. If $BD > \frac{BC}{3}$, then reflect in AD. If B' is the reflection of B', then $B'D = BD$ and the intersection of the two triangles is just ABB'. But $BB' = 2BD > \frac{2}{3}BC$, so ABB' has more than $\frac{2}{3}$ the area of ABC.

If $BD < BC/3$, then reflect in the angle bisector of C. The reflection of A' is a point on the segment BD and not D. (It lies on the line BC because we are reflecting in the angle bisector. $A'C > DC$ because $\angle CAD < \angle CDA = 90°$. Finally, $A'C \leq BC$

because we assumed $\angle B$ does not exceed $\angle A$). The intersection is just $AA'C$. But $\frac{[AA'C]}{[ABC]} = \frac{CA'}{CB} > CD/CB \geq 2/3$.

6. (PAMO 2001) Let ABC be an equilateral triangle and let P_0 be a point outside this triangle, such that $\triangle AP_0C$ is an isosceles triangle with a right angle at P_0. A grasshopper starts from P_0 and turns around the triangle as follows. From P_0 the grasshopper jumps to P_1, which is the symmetric point of P_0 with respect to A. From P_1, the grasshopper jumps to P_2, which is the symmetric point of P_1 with respect to B. Then the grasshopper jumps to P_3 which is the symmetric point of P_2 with respect to C, and so on. Compare the distance P_0P_1 and P_0P_n. $n \in N$.

Solution: We can use coordinate geometry to solve the problem. Let $P_0 = (0,0)$, $A = (0,a)$, and $C = (a,0)$, making $AC = a\sqrt{2}$. To calculate the coordinates of B, note that $BP_0 \perp AC$ since BCP_0A is a kite. Thus, BP_0 bissects AC, so $BP_0 = \frac{a\sqrt{2}}{2} + \frac{a\sqrt{6}}{2}$. Additionally, $\angle BP_0C = 45°$ because $\angle BP_0C$ bissects $\angle AP_0C$. Thus, the coordinates of B are $(\frac{a+a\sqrt{3}}{2}, \frac{a+a\sqrt{3}}{2})$.



By repeatedly applying the Midpoint Formula, we can determine the coordinates of P_1, P_2, P_3, and so on. We can also use the Distance Formula to calculate the distance of P_0P_1, P_0P_2, and so on. The values are shown in the below table.

n	Coordinates of P_n	P_0P_n
1	$(0,2a)$	$2a$
2	$(a+a\sqrt{3}, -a+a\sqrt{3})$	$2a\sqrt{2}$
3	$(a-a\sqrt{3}, a-a\sqrt{3})$	$a\sqrt{6}-a\sqrt{2}$
4	$(-a+a\sqrt{3}, a+a\sqrt{3})$	$2a\sqrt{2}$
5	$(2a,0)$	$2a$
6	$(0,0)$	0
7	$(0,2a)$	$2a$

Note that the coordinates of P_n as well as the distance P_0P_n cycle after $n = 6$. Thus, $P_0P_n = \frac{\sqrt{6}-\sqrt{2}}{2} \cdot P_0P_1$ if $n \equiv 3 \pmod 6$, $P_0P_n = 0$ if $n \equiv 0 \pmod 6$, $P_0P_n = P_0P_1$ if $n \equiv 1, 5 \pmod 6$, and $P_0P_n = \sqrt{2} \cdot P_0P_1$ if $n \equiv 2, 4 \pmod 6$.