UBC Math Circle 2020 Problem Set 2

Problems will be ordered roughly in increasing difficulty

1. A string is cut in two pieces at a point selected uniformly at random. What is the probability that the longer piece is at least x times as large as the shorter piece?

> **Solution:** Assume that $x \geq 1$. If one piece is x times longer than the other, then the cut must have happened in the first $\frac{1}{x+1}$ or the last $\frac{1}{x+1}$ of the string. These regions are disjoint, so this occurs with probability $\frac{2}{x+1}$.

2. In the hat problem, n people are each assigned, uniformly at random, a black or white hat. Everyone can see all hats, except for their own.

   Each person is asked to guess the colour of their own hat. They may guess black, white, or choose to abstain. No communication is allowed, except for a general discussion of strategy before the game begins.

   All choices (between guessing black, white, or abstaining) must be made simultaneously. The team wins if at least one person guesses correctly, and no one guesses incorrectly. Otherwise, the team loses.

   For example, if each person chooses between guessing black, white, or abstaining with equal likelihood, the probability that the team wins is $(\frac{2}{3})^n - (\frac{1}{3})^n$. (Why?)

   Find, with proof, an optimal strategy for $n = 3$. Bonus: investigate this problem for other values of n.

> **Solution:** Perhaps surprisingly, we can do better than the naive strategy of having one person guess at random and everyone else abstaining (which wins with probability $\frac{1}{2}$).
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> Consider the following strategy: each person looks at the other two hats. If they are the same colour, the person should guess the opposite colour. If the hats are two different colours, the player should abstain from guessing. Casework and computation shows that the probability of the team winning is $\frac{3}{4}$.
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> To prove optimality, we note that in any strategy, the probability P_i of player i guessing correctly is the same as the probability of player i guessing incorrectly. Since at least one player must guess correctly for the team to win, the probability P of the team winning is $\leq \sum_{i=1}^{n} P_i$. Since any player guessing incorrectly will cause the team to lose, the probability $1 - P$ of the team losing is $\geq$ each P_i.
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> Combining these inequalities, we find that $P \leq \sum_{i=1}^{n} P_i \leq n(1-P)$, and so $P \leq \frac{n}{n+1}$. For $n = 3$, our strategy attains the upper bound of $\frac{3}{4}$.

Bonus: the hat problem is related to the study of *covering codes*. For the special cases $n = 2^k - 1$ or $n = 2^k$, an optimal strategy is known using *Hamming codes* and *extended Hamming codes*, respectively.

3. 2020 people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a seat, chosen uniformly at random. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats otherwise, chosen uniformly at random. What is the probability that the last person to board gets to sit in his assigned seat?

Solution: Instead of the thinking of each person boarding the plane and moving to a random unoccupied seat if his assigned seat is taken, think of the person evicting whoever was in the seat and taking back his seat. This means that the first unfortunate passenger will get repeatedly evicted. By the time the last person to board will take his seat, the first boarder will either be in the correct seat, or the last person's seat. To the first boarder, both seats appear identical, so both scenarios happen with probability $\frac{1}{2}$. Hence the probability the last person sits in his assigned seat is exactly $\frac{1}{2}$.

4. A certain store sells specialty Pokémon plushies. They have n unique plushies in stock. Every day, they pick one plushie uniformly at random and put it on display. What is the expected number of days required so that every plushie will have been on display at least once?

Solution: This problem is also known as the coupon collector's problem.

Let T be the time it takes to have each plushie be on display at least once, and t_i be the time it takes to get the ith unique plushie, given that $i - 1$ unique plushies have been put on display. Note that the probability to get the ith unique plushie on the day is $p_i = \frac{n-i+1}{n}$. To calculate the expected value of t_i we notice that if we fail to get a unique plushie, we need to wait one more day and wait the expected amount of time to get the ith unique plushie again. Formally:

$$\mathbb{E}[t_i] = p_i + (1 - p_i)(1 + \mathbb{E}[t_i])$$

Rearranging the equality and solving for $\mathbb{E}[t_i]$ gives us what we want:

$$\mathbb{E}[t_i] = \frac{1}{p_i}$$

We need to wait to get the ith coupon for all i from 1 to n, so by linearity of expectation, we get:

$$\mathbb{E}[T] = \mathbb{E}\left[\sum_{i=1}^{n} t_i\right] = \sum_{i=1}^{n} \mathbb{E}[t_i] = \sum_{i=1}^{n} \frac{1}{p_i} = n\left(\sum_{i=1}^{n} \frac{1}{n-i+1}\right) = n\left(\sum_{i=1}^{n} \frac{1}{i}\right) \approx n \log n$$

The last approximation can be made since the harmonic sum up to n is approximately $\log n$. This can be proved via integrals.

5. You are given an unfair coin with probability of heads $p \in (0,1)$. Design an experiment with two outcomes, A and B such that $P(A) = 0.5 = P(B)$. Bonus: Design an experiment such that $P(A) = x$ and $P(B) = 1 - x$ for any $x \in [0,1]$.

Solution: Toss the coin until it comes up heads. Let N_1 be the number of tosses required to achieve this. Repeat, letting N_2 be the number of tosses required for the second heads. If $N_1 = N_2$, start over. Otherwise, $N_1 < N_2$ with probability $\frac{1}{2}$.

An alternate way is to toss the coin twice. If the two flips are the same, repeat. Once you have two different flips, the outcomes HT and TH are equally likely.

For the bonus, first apply the previous part to assume WLOG that the coin is fair. Now consider a binary expansion of $x = 0.x_1x_2x_3.....$ Flip the fair coin until it comes up heads, and let N be the number of flips required. Let outcome A be the probability that $x_N = 1$. This happens with probability

$$P(N=1)x_1 + P(N=2)x_2 + ... = \frac{1}{2}x_1 + \frac{1}{2^2}x_2 + ... = 0.x_1x_2x_3$$

6. (Putnam 1993 B-3) Two real numbers X and Y are chosen uniformly at random in the interval $(0,1)$. Compute the probability that the closest integer to $\frac{X}{Y}$ is even. If possible, express the answer as $r + s\pi$ where $r, s \in \mathbb{Q}$. Otherwise, leave it as an infinite series.

Solution: Let N be the closest integer to $\frac{X}{Y}$. Then $N = 0$ if $2X < Y$. This has probability of $\frac{1}{4}$. To see this, one can draw a square, compute an integral, or observe the following: The probability of $X < Y$ is $\frac{1}{2}$. Assume this is the case. Then we can assume WLOG X is distributed uniformly from 0 to Y. Then $X < \frac{Y}{2}$ happens with probability $\frac{1}{2}$, thus the total probability is $\frac{1}{4}$. Next, we compute the probability that $2n - \frac{1}{2} < \frac{x}{y} < 2n + \frac{1}{2}$, which, by similar means, works out to be $\frac{1}{4n-1} - \frac{1}{4n+1}$. Then the total probability is $\frac{1}{4} + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{7} - \frac{1}{9}) + ... = \frac{5-\pi}{4}$.