UBC Math Circle 2020 Problem Set 4

Problems will be ordered roughly in increasing difficulty

1. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner and always moves to an adjacent subcube (sharing a face of area 1) can it do this and eat the centre subcube last? If yes then give an order of subcubes eaten, if no then prove it is impossible.

Solution: One colours each subcube (i, j, k) white if $i + j + k \equiv 0 \pmod 2$ and black if otherwise. Note that for each movement that the mouse makes, the colour of its subcube alternates. However, since the centre subcube and any corner are coloured differently, the mouse has to traverse an even number of subcubes which is a contradiction since there are 27 subcubes.

2. Define a trail of an undirected graph G to be a sequence of connecting edges such that no edges occur more than once. Let G be a connected graph and let k be the length of the longest trail. Prove that if 2 trails in G have length k, then there must be some vertex $v \in V(G)$ that is contained in both trails.

Solution: Let T_1 and T_2 be the two trails. Because G is connected, there must be a path from vertex $v_1 \in T_1$ to some $v_2 \in T_2$. WLOG, we may assume that v_1 is the last time that this path intersects T_1, and v_2 is the first time it intersects T_2. We observe that v_1 (resp v_2) divides T_1 (resp T_2) into two parts, one of which must have length $\geq \frac{k}{2}$. Consider the trail that takes each longer half, as well as the path connecting v_1 to v_2. By construction, if $v_1 \neq v_2$, there must be at least one edge between v_1 and v_2, so our new path has length $\geq \frac{k}{2} + \frac{k}{2} + 1 = k + 1$, a contradiction.

3. Suppose there are n people at a meeting, and every group of four people contains a person who know the other three. Prove that there is a person who knows every other person at the meeting.

Solution: This problem only makes sense for $n \geq 4$. Let v be a vertex of maximum degree. We wish to show that every other vertex is adjacent to v. Suppose not, so there exists w that is not adjacent to v. For any pair of vertices x, y adjacent to v, consider the set of 4 vertices $\{v, w, x, y\}$. We have that v and w are not adjacent, so one of x, y (WLOG x) realizes the given property. In particular, x and y are always adjacent. Thus, we have x is adjacent to every other neighbour of v, as well as v and w, giving x higher degree than v.

4. (BWM 1998) There exist polyhedrons that have more faces than vertices. Find the smallest number of triangular faces that such a polyhedron can have.

> **Solution:** The minimum number of triangles in such a polygon is 6. One such construction can be obtained by gluing two tetrahedrons together by the face.
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> We can use Euler's characteristic $V - E + F = 2$, as any polygon can be thought of as a planar graph. Let T denote the number of triangular faces of a polyhedron, Q be the number of quadrilateral faces and X be the number of faces with 5 or more sides. By assumption $F > V$ so $2F > 2 + E$. Furthermore, $F = T + Q + X$ and summing up the edges along each face we get $2E \geq 3T + 4Q + 5X$. Hence,
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> $$4F = 4(T + Q + X) > 4 + 2E \geq 4 + 3T + 4Q + 5X$$
>
> Rearranging the terms we get:
> $$T > 4 + X$$
>
> Clearly $T > 4$. Can we have a polygon one where $T = 5$? Then $X = 0$ and Q is arbitrary. We can't because the sum of edges around the faces must be even, so we conclude that $T \geq 6$.

5. Let Q_k be the graph whose vertices correspond to the sequences $(a_1, a_2, \cdots, a_k)$ where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. Show that we can traverse from some vertex, visiting each once and going back to the initial vertex (Hamiltonian).

> **Solution:** We can construct this Hamiltonian cycle inductively. For $k = 0$, the empty path suffices in the empty Q_0 graph.
>
> Let's denote the vertices of the Hamiltonian cycle of Q_k as the list ordered in the order of the cycle $H_k = \{v_1, v_2, v_3, \ldots v_{2^k}\}$, and $\overline{H_k} = \{v_{2^k}, v_{2^k-1}, \ldots, v_1\}$ denote the reversed set. Let $(0H_k)$ denote the sequence where you replace every instance of v_i with $(0, v_i)$, and similarly for $(1H_k)$. Then, a Hamiltonian cycle for $k + 1$ can be written concisely as:
> $$H_{k+1} = (0H_k, 1\overline{H_k})$$
> This construction is called a Gray code.

6. (JMO 1997) Let G be a graph with 9 vertices. Suppose given any five points of G, there exist at least two edges with both endpoints among the five points. What is the minimum possible number of edges in G?

Solution: The minimum number of edges 9, with three disjoint cycles of size 3.

Let a_n be the minimum number of edges in a graph with n vertices satisfying the condition. We can show that $a_{n+1} \geq \frac{n+1}{n-1}a_n$. Consider any graph satisfying this property on $n+1$ vertices and let l_i denote the number of edges left in the graph if we removed vertex i and all edges adjacent to it. Clearly $a_n \geq l_i$ for all i. Furthermore $l_1 + l_2 + \cdots + l_{n+1} = (n-1)a_{n+1}$, since each edge is counted $n-1$ times when we sum up the edges from these graphs.

Since $a_5 = 2$, $a_6 \geq 3$, $a_7 \geq 5$, $a_8 \geq 7$, we get $a_9 \geq 9$.