UBC Math Circle 2020 Problem Set 5

Problems will be ordered roughly in increasing difficulty

1. On a chalkboard are written the numbers from 1 to 2020. Two players take turns erasing a number from the board. The game ends when two numbers remain: the first player wins if the sum of these numbers is divisible by 3, the second player wins otherwise. Which player has a winning strategy?

Solution: The second player does. If the first player chooses x the second player chooses $2021 - x$. In the end, the last two numbers will sum up to 2021 which is not divisible by 3.

2. (1996 Tournaments of the Towns) In a lottery game, a person must select six distinct numbers from $1, 2, 3, \ldots, 36$ to put on a ticket. The lottery committee will then draw six distinct numbers randomly from $1, 2, 3, \ldots, 36$. Any ticket with numbers not containing any of these six numbers is a winning ticket. Show that there is a scheme of buying 9 tickets guaranteeing at least a winning ticket, but 8 tickets is not enough to guarantee a winning ticket in general.

Solution: One solution:

$$(1, 2, 3, 4, 5, 6), (1, 2, 3, 7, 8, 9), (4, 5, 6, 7, 8, 9)$$

$$(10, 11, 12, 13, 14, 15), (10, 11, 12, 16, 17, 18), (13, 14, 15, 16, 17, 18)$$

$$(19, 20, 21, 22, 23, 24), (25, 26, 27, 28, 29, 30), (31, 32, 33, 34, 35, 36)$$

If first three aren't winning, the committee must have drawn two numbers from among them. Same with the next three tickets. For the last three tickets, the committee needs to draw one number from each ticket, but then it would need to have drawn 7 total numbers.

Now to prove that 8 tickets aren't enough. Observe that if any one number appears on more than 3 tickets, then there would only be 5 tickets left, one for each remaining number the committee may draw.

On the other hand, as there are 48 numbers on 8 tickets among 36 numbers, some number must appear on two of the tickets. This leaves us with 36 numbers on 6 tickets among 35 numbers since we removed a number. Hence there must be some other number that appears on two tickets. Now we're left with four tickets, and four numbers left to pick, so we can't win with any set of 8 tickets.

3. (USAMO 1999) The board consists of a row of n squares, initially empty. Players take turns selecting an empty square and writing either an S or an O in it. The player who first succeeds in completing SOS in consecutive squares wins the game. If the whole board gets filled up without an SOS appearing consecutively anywhere, the game is a draw.

   (a) Suppose $n = 4$ and the first player puts an S in the first square. Show the second player can win.

   (b) Prove that for $n = 4$ neither player has a winning strategy.

   (c) Show that the first player has a winning strategy for $n = 7$.

   (d) Show that the second player has a winning strategy for $n = 2020$

Solution:

(a) The second player (P2) puts an S in the last square. Then if the first player (P1) puts down an O, it is adjacent to and S and there is an empty spot to complete SOS. Otherwise, they put down an S, and there is a gap of one space to another S.

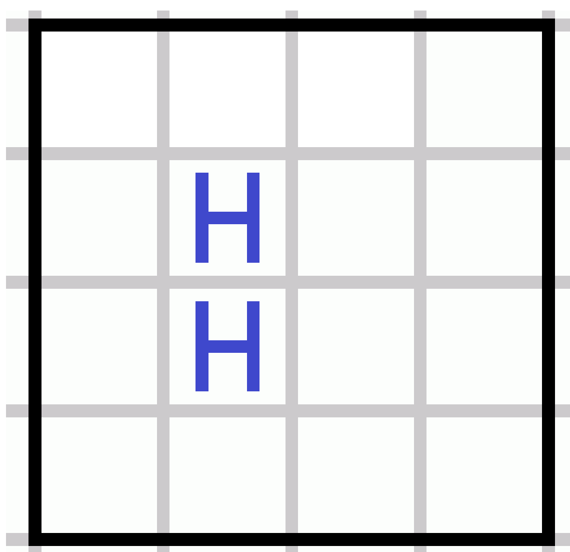
(b) Part (a) shows that P1 cannot put down an S in the first square. Thus we have 3 cases left to check.
1. If P1 puts down an O in the first square, P2 can force a tie by putting down an O in the second square. If P2 plays anywhere else, P1 can force a tie by playing an O in the second square. Otherwise, P2 puts down an S in the second square, and P1 can force a tie by putting down an S in the third square.
2. If P1 puts down an S in the second square, P2 can force a tie by putting down an S in the third square. If they move anywhere else, P1 can force a tie by putting an S in the third square. P2 would not put an O in the third square because that gives P1 the win.
3. If P1 puts down an O in the second square, P2 can force a tie by putting an O in the first square. If P2 moves anywhere else, P1 can force a tie by putting an O in the first square. P2 would not put down an S in the first square, because that would give P1 the win.

(c) P1 puts down an S in the middle square. P2 plays on one of the halves, WLOG, in positions $1 - 3$. If P1 can complete SOS, they do. Otherwise the spots $4 - 7$ are the case of part (a), but P1 now has the position of P2, and plays an S in the last position. If P2 instead decides to place one more letter in positions $1 - 3$, there will still be one last spot left in that range, that P1 can play in.

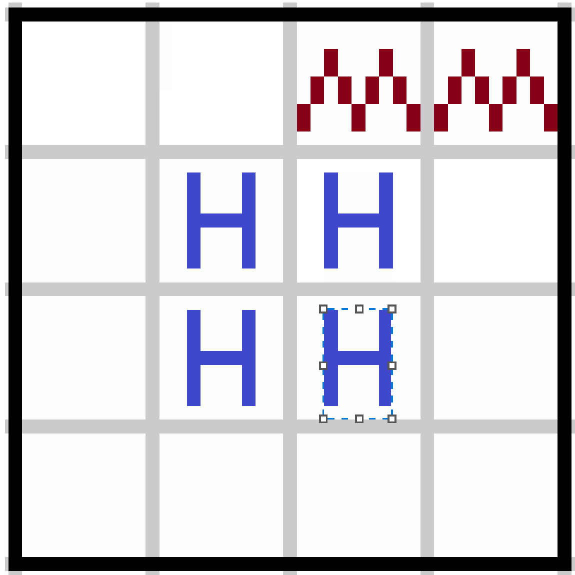
(d) We employ a similar strategy to part (c). Wherever P1 plays, P2 puts down an S at distance 4 from P1's letter and from either edge. P1 may put down a letter wherever, WLOG to the left of P2's S. If P2 can complete SOS at this point, they do. Otherwise, they play an S at a distance 4 to the right of their first S. These 4 cells are the case of part (a), with second player having the winning strategy in those 4 cells. Furthermore, there is an even number of squares remaining, so that the parity will not shift by moving outside these four cells. Compare this to part (c). If P1 employs the same strategy they did in part (c), there would be an odd number of spaces left at the end of their turn, meaning there would be a parity shift.

4. Henry and Morgan are playing Domineering on a 4x4 board. The rules are as follows: Henry can place 1x2 dominoes (vertical) on a valid empty spot. Morgan can place 2x1 dominoes (horizontal) on a valid empty spot. The winner is the last player to make a valid move. Determine which player has a winning strategy if Henry starts.

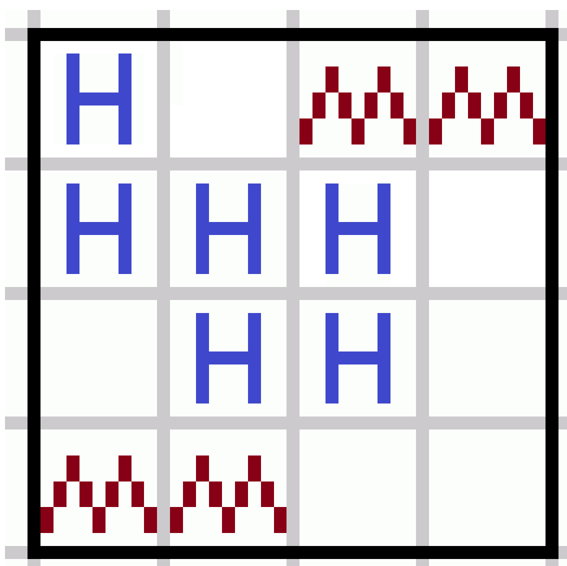
Solution: Henry has a winning strategy. He places a domino in the center 4 squares, as shown:



If Morgan does not place a domino on the remaining center two squares, Henry does, and we have a situation that looks sort of like below:

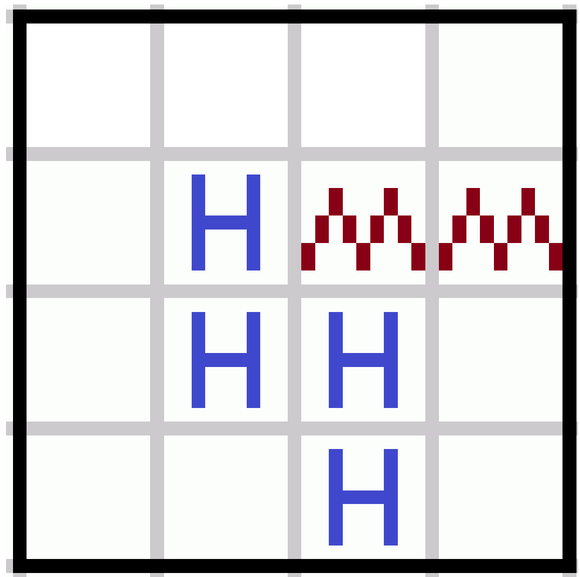


At this point, it is Morgan's turn, and he has at most three valid moves left, and Henry will always have at least two valid moves. However, no matter what Morgan does, Henry can place a domino to screw up one of Morgan's remaining moves, such as below:

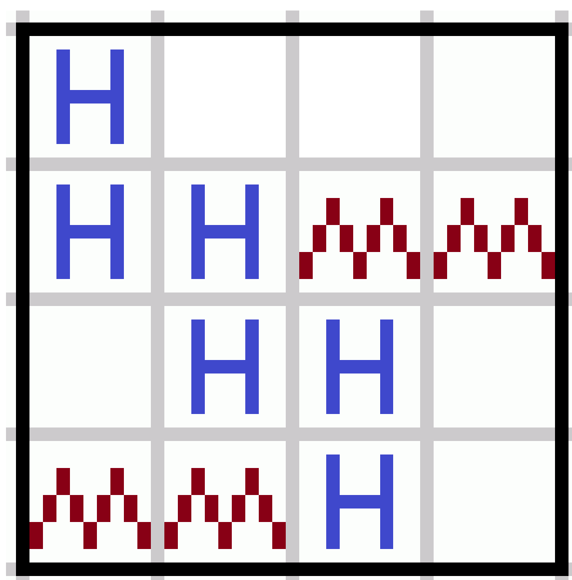


Thus it is Morgan's turn, and both players have one valid move left.

Thus Morgan must play in one of the center squares, and Henry responds by taking the remaining center square. WLOG the position looks like this:



At this point we are in a similar position to the previous case: It is Morgan's turn, and he has at most three valid moves left, while Henry has at least two. However, no matter where Morgan plays, Henry can screw up one of Morgan's remaining moves, such as below:

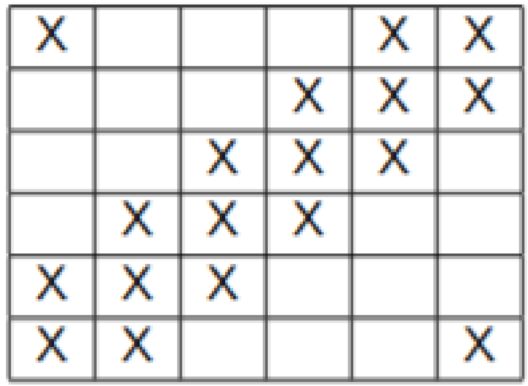


At this point it is clear again that Henry wins.

5. (USAMO 2004) Tak and Tam are playing a game. They take turns writing distinct (real) numbers in a 6x6 grid. Once the grid is filled up, the largest number in each row is coloured in. Tak wins if there is a straight line connecting opposite sides. Tam wins

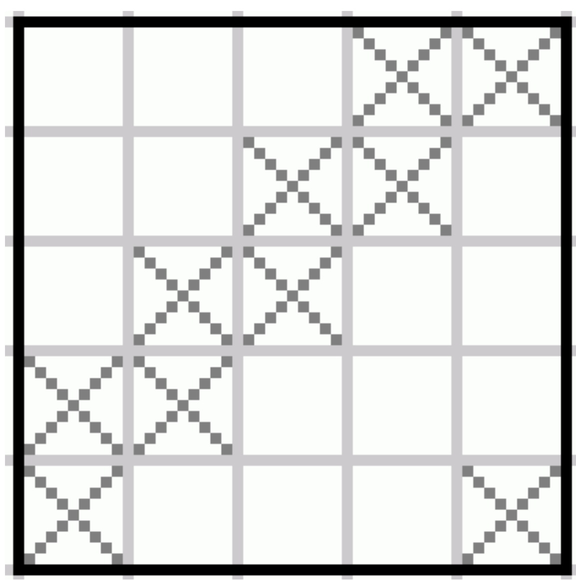
otherwise. Tak starts. Determine which player has a winning strategy, and describe it, or prove that neither does. Bonus: Do the same for a 5x5 grid.

Solution: Bob selects three squares in each row as follows:



Whenever Alice writes a number in a marked square, Bob writes a larger number on an unmarked square in the same row. Whenever Alice writes a number on an unmarked square, Bob writes a smaller number on a marked square in the same row. In this way, no selected squared can be black, and so no black line is possible.

For the 5×5 case, Bob can play a similar game, selecting the following 10 squares:



It does not matter that there are extra unselected squares, as this strategy just requires less than half the squares in a row to be selected. Thus none of the selected squares can be black, and there is no line possible.

6. (St. Petersburg City MO) Two players play the following game on a 100×100 board. The first player marks a free square, then the second player puts a 1×2 domino down covering two free squares, one of which is marked. This continues until one player is unable to move. The first player wins if the entire board is covered, otherwise the second player wins. Which player has a winning strategy?

Solution: The first player has a winning strategy. Let us say a position is stable if every square below or to the right of a free square is free. Then we claim the first player can always ensure that on his turn, either the position is stable or there is a free square with exactly one free neighbor (or both).

Let us label the square in the i-th row and j-th column as (i, j), with $(1, 1)$ in the top left. We call a free square a corner if is not below or to the right of another free square. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ be the corners from top to bottom.

First notice that if (a, b) is a corner such that both $(a + 1, b - 1)$ and $(a - 1, b + 1)$ are nonfree (or off the board), then the first player may mark (a, b), and however the second player moves, the result will be a stable position. More generally, if $(a, b), (a+1, b-1), \ldots, (a+k, b-k)$ are corners and $(a-1, b+1)$ and $(a+k+1, b-k-1)$ are both nonfree or off the board, the first player can be sure to return to a stable position.

To show this, first note that we cannot have both $a = 1$ and $b - k = 1$, or else the number of nonfree squares would be odd, which is impossible. Without loss of generality, assume that $b - k \neq 1$ is not the final corner. The first player now marks (a, b). If the second player covers (a, b) and $(a, b + 1)$, the position is again stable. Otherwise, the first player marks $(a + 1, b - 1)$ and the second player is forced to cover it and $(a + 2, b - 1)$. Then the first player marks $(a + 2, b - 2)$ and the second player is forced to cover it and $(a+3, b-2)$, and so on. After $(a+k, b-k)$ is marked, the result is a stable position. (Note that the assumption $b - k \neq 1$ ensures that the moves described do not cross the edge of the board.)

To finish the proof, we need to show that such a chain of corners must exist. Write the labels $(a_1, b_1), \ldots, (a_k, b_k)$ in a row, and join two adjacent labels by a segment if they are of the form $(a, b), (a+1, b-1)$. If two adjacent labels $(a, b), (a+i, b-j)$ are not joined by a segment, then either $i = 1$ or $j = 1$ but not both. If $i = 1$, draw an arrow between the labels pointing towards $(a + i, b - j)$; otherwise draw the arrow the other way. Also draw arrows pointing to (a_1, b_1) and (a_k, b_k).

There is now one more chain of corners (joined by segments) than arrows, so some chain has two arrows pointing to it. That chain satisfies the condition above, so the first player can use it to create another stable position. Consequently, the first player can ensure victory.