UBC Math Circle 2021 Problem Set 2

1. (a) Let *Stirling numbers of the second kind* be denoted by $\left\{ {n \atop k} \right\}$, which counts the number of ways to partition a set of n elements into k nonempty subsets. Prove that they satisfy the recurrence

$$\left\{ {n+1 \atop k+1} \right\} = (k+1)\left\{ {n \atop k+1} \right\} + \left\{ {n \atop k} \right\}.$$

   (b) Prove for all n and all $x \in \mathbb{R}$ that

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)...(x-k+1).$$

   You may or may not want to use part (a).

2. (CMO 2015) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every $n \in \mathbb{N}$.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(a)) = a + 1$ for all $a \in \mathbb{Z}$.

4. (a) Let $a, b, c \in \mathbb{Z}$ be coprime to 13. Show that $a^3 + b^3 + c^3$ is also coprime to 13.

   (b) Does there exist a configuration of 13 people such that in every group of 3 there is a pair of mutual strangers, and in every group of 5 there is a pair of mutual friends?

5. There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that if $n \equiv 1 \pmod 3$ its impossible to reach a state with only two markers remaining.

6. Let $x, y \in \mathbb{R}$ satisfy $x + y = 1$. Show for all $m, n \in \mathbb{N}$ that

$$x^{n+1} \sum_{j=0}^{m} \binom{n+j}{j} y^j + y^{m+1} \sum_{j=0}^{n} \binom{m+j}{j} x^j = 1.$$