UBC Math Circle 2021 Problem Set 5

1. Let $P(x)$ be a real polynomial such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist real polynomials $f(x)$ and $g(x)$ such that $P(x) = f(x)^2 + g(x)^2$.

2. Prove or disprove: for every $k \geq 1$, if $\mathbb{N}$ is coloured with k colours, then there must exist a monochromatic triple $(x, y, z) \in \mathbb{N}^3$ satisfying
$$x + y = 3z.$$

3. Let five points on a circle be labelled A, B, C, D, E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$. Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

4. Do there exist two weighted dice (with faces numbered from 1 to 6) such that the sum of the dice in a random roll is uniformly distributed in $\{2, 3, \ldots, 12\}$?

5. A *partition* of n is a weakly-sorted list of positive integers $(\lambda_1, \ldots, \lambda_\ell)$ whose sum is n. Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

6. Let p be a prime. Show that $x^2 + x + 1 \equiv 0 \pmod{p}$ has a solution iff $p \equiv 1 \pmod{3}$.