UBC Math Circle 2021 Problem Set 6

1. A fair coin is flipped until a heads appears. What is the expected number of flips needed?

2. In the triangle $\triangle ABC$, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

3. Find all integers n for which $n^3 + 23$ is a perfect square.

4. Show that there exists three consecutive vertices A, B, and C in every convex n-gon (with $n \geq 3$) such that the circumcircle of $\triangle ABC$ covers the entire n-gon.

5. (Sylvester-Gallai) Let S be a finite set of points in the plane that do not all lie on the same line. Show that there exists a line passing through exactly two points of S.

6. (a) Determine all $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(xy)$.
   (b) Determine all $f : \mathbb{N} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$.
   (c) Determine all $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$.
   (d) Determine all $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$.

   Note that there are functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x)+f(y)$ that don't take the form $f(x) = cx$. These are pathogogical, can't be described explicitly, and require the axiom of choice (specifically, a Hamel basis) to construct non-explicitly.

7. (Frobenius coin problem) The motivation is thus: given some denominations (values) of coins, we wish to determine which amounts are obtainable. For example, with coins of value 2 and 4, the obtainable amounts are the even natural numbers. With values of 2 and 5 instead, we can obtain every natural number except 1 and 3. Let a, b be coprime natural numbers. We define the Frobenius number, $F(a, b)$, to be the largest integer which cannot be written as a non-negative linear combination of a and b, ie the largest integer that can't be written as $ax + by$ for $x, y \in \mathbb{Z}_{\geq 0}$

   (a) Determine $F(2, b)$.
   (b) Determine $F(3, b)$.
   (c) Determine $F(a, b)$.

   Remark: It is possible to ask the same question but with any number of different coin values. The formulae for $F(x_1, \ldots, x_n)$ are known for at least $n = 3$, and possibly larger n.