UBC Math Circle 2021 Problem Set 7

1. Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2021 \pmod{p}.$$

2. Draw a regular n-gon with sidelength 1. What is the sum of squared distances from one vertex to every other vertex?

3. Show that for any non-negative integer n, we have

$$4^n = \sum_{j=0}^{n} 2^{n-j} \binom{n+j}{j}.$$

4. (a) An olympiad had six problems and 200 contestants. The contestants are very skilled, so each problem is solved by at least 120 of the contestants. Prove that there exist two contestants such that each problem is solved by at least one of them.

   (b) Find, with proof, the least positive integer n such that if each problem is solved by at least n of the contestants, then there exist two contestants such that each problem is solved by at least one of them.

5. A polynomial $P(x)$ of degree 2020 satisfies $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, 2020$. Find $P(2021)$.

6. How many pairs of integers (m, n) with $1 \leq m, n \leq 2021$ satisfy

$$(m^2 - mn - n^2)^2 = 1?$$

7. Let $n > 1$ be an integer and let $a > 0$ be a real number. Let $x_1, \ldots, x_n$ be non-negative real numbers satisfying $\sum_{i=1}^{n} x_i = a$. Find the maximum of

$$\sum_{i=1}^{n-1} x_i x_{i+1}.$$