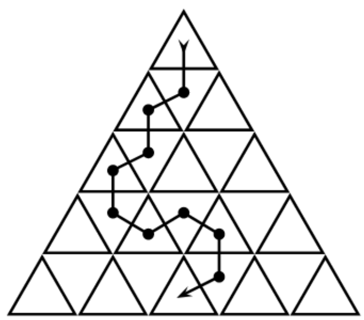
UBC Math Circle 2021 Problem Set 8

1. Find all subsets $A \subset \mathbb{Z}$ such that $\{x + y : x \in A, y \in \mathbb{Z} \setminus A\} \subset A$.

2. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + c^x + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

3. An organization has n members, and it has $n + 1$ three-member committees, no two of which have identical membership. Prove that there are two committees that share exactly one member.

4. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2021)$.



5. Consider n real numbers, not all zero, with sum zero. Prove that one can label the numbers as $a_1, a_2, \ldots, a_n$ such that $a_1a_2 + a_2a_3 + \cdots + a_na_1 < 0$.