UBC Math Circle 2021 Problem Set 9

1. (AIME 2009 II) Let $x_1, x_2, x_3, x_4, x_5, x_6$ be nonnegative real numbers such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \quad \text{and} \quad x_1x_3x_5 + x_2x_4x_6 \geq \frac{1}{54}.$$

   Find the maximum possible value of

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2.$$

2. (AIME 2008 II) Let a and b be positive real numbers. Suppose that the following system of equations

$$a^2 + y^2 = b^2 + x^2 = (a - x)^2 + (b - y)^2$$

   has a solution in (x, y) satisfying $0 \leq x < a$ and $0 \leq y < b$. Find the maximum possible value of a/b.

3. Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be positive integers with $a_1 \leq a_2 \leq \cdots \leq a_n$. Suppose that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1.$$

   Show that $a_n < 2^{n!}$.

4. Let $f(x)$ and $g(x)$ be nonzero polynomials with $f(x^2 + x + 1) = f(x)g(x)$. Show that f has even degree.

5. Beren and Luthien play a game on a $2n \times 2n$ grid. Initially the grid is empty. Each turn, Beren first marks an empty cell on the grid. Then, Luthien places a 1×2 domino (which she may rotate) so that the domino covers two empty cells of the grid, one of which must be the cell marked by Beren. The game ends when no more dominoes can be placed. Beren wins if the grid is covered entirely. Can Beren win?

6. Suppose there are 32 points inside a 4×4 square. Show that the distance between the closest pair of points is at most 1.