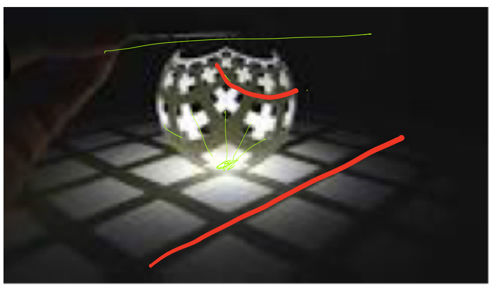
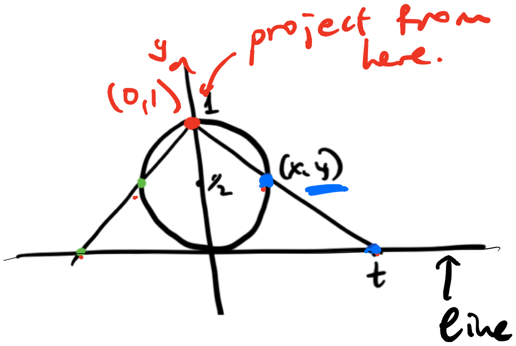
Projective geometry

A story that started 2400 years ago; connects Euclid, Pappus, Pascal, and the modern times.

Warm-up: stereographic projection.





Problem 1: find (x,y) in terms of t.

Express (x,y) in terms of t.

Hint: use similar triangles.

How to make a new mathematical object? Summary:

Axioms

projective line

"a line + a point at ∞"

"plus a point at ∞".

$x \to \infty$

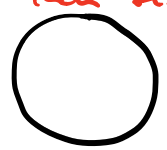
want to make sense of "$x = \infty$"

"a point at ∞"

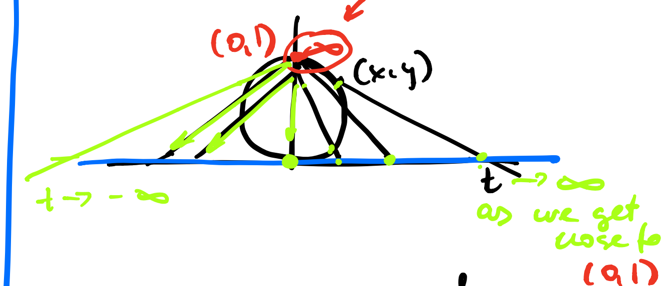
Geometric realizations

if $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x)$ then $f(x)$ extends continuosly to the proj. line

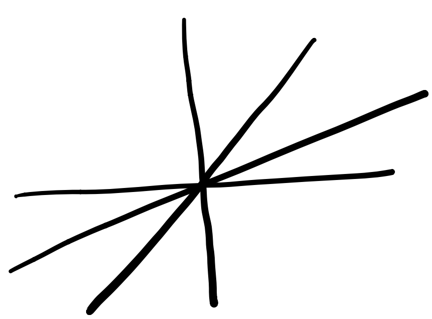
1)



circle



2) The space of lines in the plane:



each line in the plane is a "point" on the projective line

Coordinates

$(x:y)$:

$(x,y) \sim (\lambda x, \lambda y)$

projective coordinates

Now, you can take the coordinates from any field [a set with addition and multiplication]

For example, $\mathbb{P}^1(\mathbb{R})$, $\mathbb{P}^1(\mathbb{F}_5)$

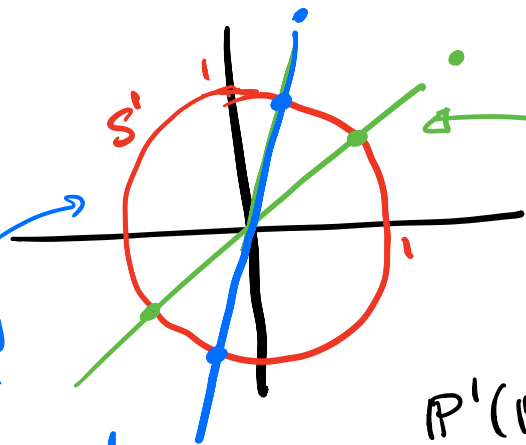
see problem set

Different construction of the proj. line

- Take something huge, identify a lot of points in it.

Start with a plane.

$\mathbb{R}^2$

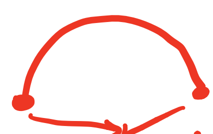


every line is just one point in our new space.

every line is determined by its intersection point with S^1
(only need the top half of S^1)

$\mathbb{P}^1(\mathbb{R})$ = the set of lines through $(0,0)$ in $\mathbb{R}^2$.

$\parallel$



glued together (both define x-axis)

Again looks like a circle.

This construction also gives us projective coordinates:

projective line

So, $\mathbb{P}^1(\mathbb{R})$ is:

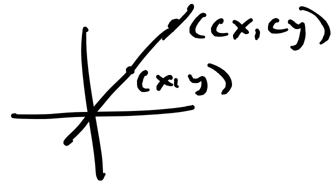
$$\{\underline{(x:y)} \mid \underbrace{x,y \in \mathbb{R}}_{\mathbb{R}^2}\}/\sim$$

$\sim$ identifies points on the same line

(you usually denote points on the plane by (x,y)

$$(x:y) \sim (cx:cy)$$

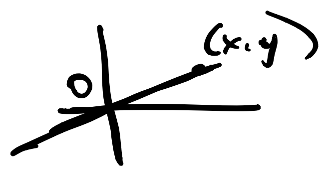
where $c \in \mathbb{R}$



point on $\mathbb{R}$ has just one coordinate;

but on $\mathbb{P}^1(\mathbb{R})$, it has two homogeneous coordinates: $(x:y)$

"Equivalence classes": one class = the whole line

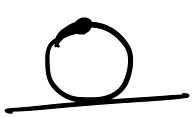


Want to find representatives: in $\mathbb{R}$

set $y=1$: $\underline{(x:y)} \sim (\frac{x}{y}, \underline{1})$

take $c = 1/y$

except if $y=0$.



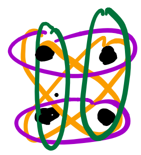
$y=0$: point at ∞

The projective plane

- A collection of points

  With a collection of selected subsets, called lines

Example:



an "affine plane over $\mathbb{F}_2$"

4 points, 6 lines

(here we have: every 2-point subset is selected)

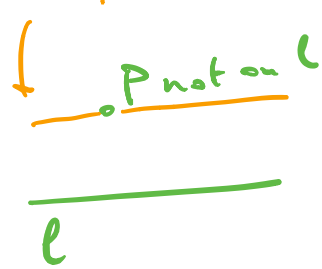
Must satisfy the axioms:

- There is exactly one line containing any given pair of points

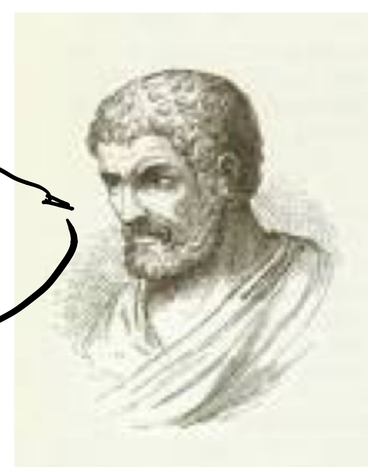
- Any two lines have a common point

Euclid's axiom that we replace with this

exists unique parallel to ℓ



what?



(4th century BC)

Note: now we have proj. coordinates $(x:y)$ on a projective line

We can make a projective line over **any** **field** (other than $\mathbb{R}$)

"collection of **numbers** s.t. you can add, subtract, multiply, divide, with usual laws of arithmetic".

examples: $\mathbb{R}$, - real

$\mathbb{Q}$ - **rational**

$\{\frac{a}{b} \mid a, b$ are integers $b \neq 0\}$.

$\mathbb{F}_p = \{0, 1, \ldots p-1\}$ - field of p elements.

p - prime

$p=2$ $\quad \mathbb{F}_2 = \{0, 1\}$.

in $\mathbb{F}_2$: $0^2 = 0 \quad 1^2 = 1$

$1+1 = 2 = 0$

$\mathbb{F}_3 = \{0, 1, 2\}$

$1+2 = 3 = 0$

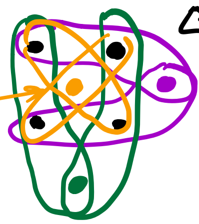
$2 \cdot 2 = 4 = 1$ in $\mathbb{F}_3$.

in $\mathbb{F}_3$, $1/2 = 2$!

Realizations

$\mathbb{P}^1(\mathbb{F}_2)$

1) For our "plane" of 4 points:



add intersection points of our "lines"

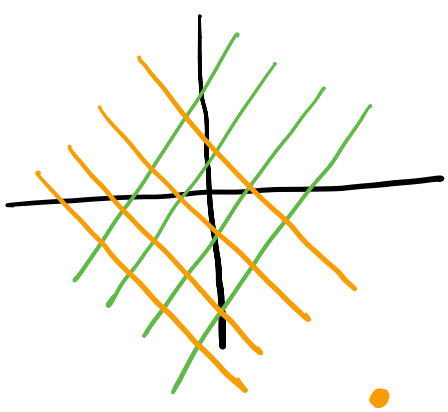
we added the intersection points of all pairs of lines

Problem: check that this satisfies the new axioms. #2

This is the smallest projective plane!

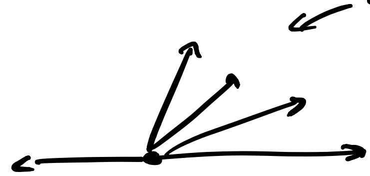
(Exer: assign projective coordinates to every point!)

2) What about the real projective plane?



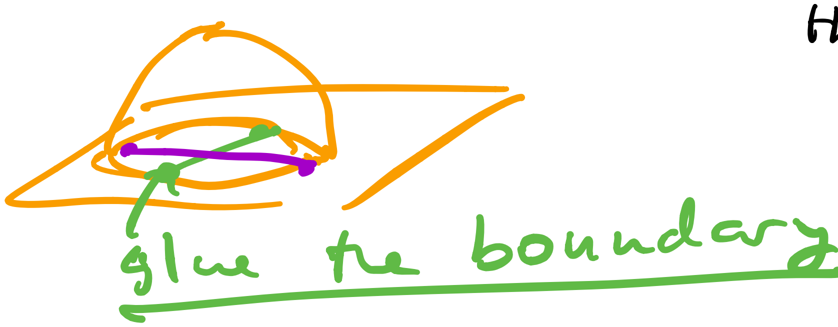
• ← add "a point at ∞" for every family of parallel lines on the plane.

What would that look like?



need to add a point at ∞ in every direction!

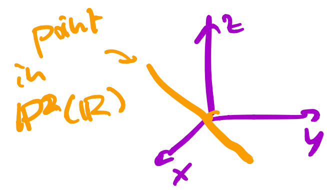
It is hard to imagine what we get: take a hemisphere and glue opposite points on the boundary.



Fact You cannot do it within our usual 3-space.

Better construction:

Make every line through (0,0,0) in $\mathbb{R}^3$ (space (3-dim)) a point in this projective plane.

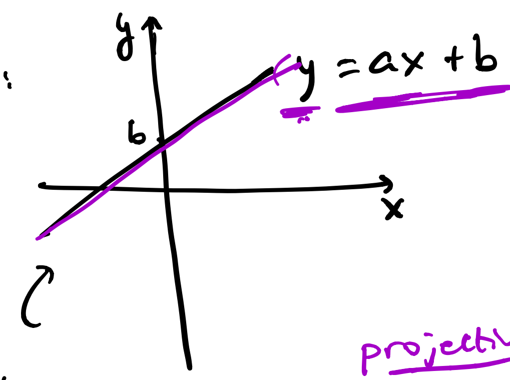


In coordinates: $(x : y : z)$

$(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for any $\lambda \in \mathbb{R}$.

What happens to some familiar equations?

- lines:



$(x : y : z) \rightsquigarrow \left(\frac{x}{z}, \frac{y}{z}\right)$ familiar coordinates

projective $z = 0$ – "line at ∞"

the line connecting $(0, b)$ with the point at ∞ given by a.

Its equation: $(x : y : z)$ such that

$$\frac{y - b}{a} = x$$

$$y - b = ax$$

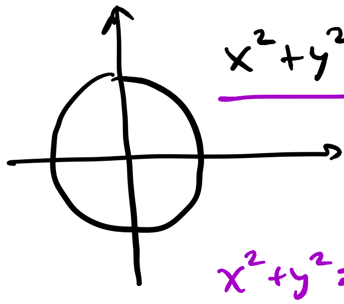
$$y - bz = ax$$

$$\frac{y}{z} = a\frac{x}{z} + b$$

$$\boxed{y = ax + bz}$$

given by homogeneous linear equations

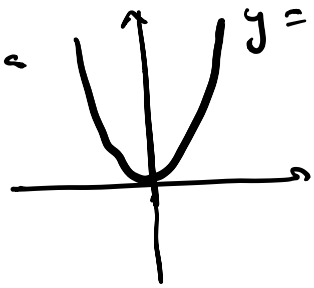
- (circles), ellipses, parabolas, hyperbolas:



$x^2+y^2=1$ – special case of an ellipse $a^2x^2+b^2y^2=1$

$x^2+y^2=z^2$ ← projective equation of a circle

- Parabola

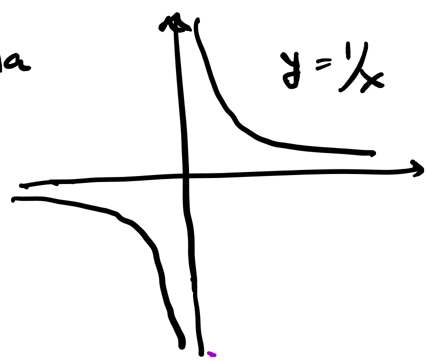


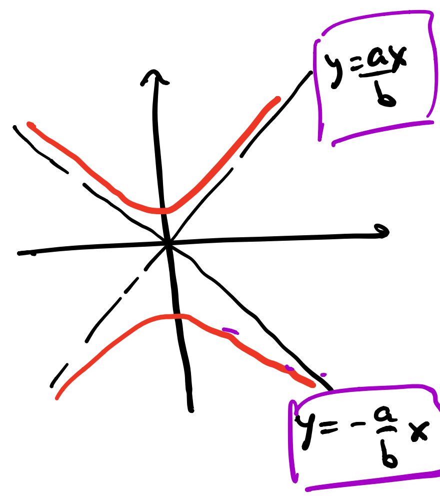
$y=ax^2$

becomes

$zy=ax^2$

- hyperbola





asymptotes are:
$x=0$
$y=0$

$xy=1$

becomes

$xy=z^2$

$$\left(y-\frac{a}{b}x\right)\left(y+\frac{a}{b}x\right)=1$$

$$y^2-\frac{a^2}{b^2}x^2=1$$

$$b^2y^2-a^2x^2=1$$

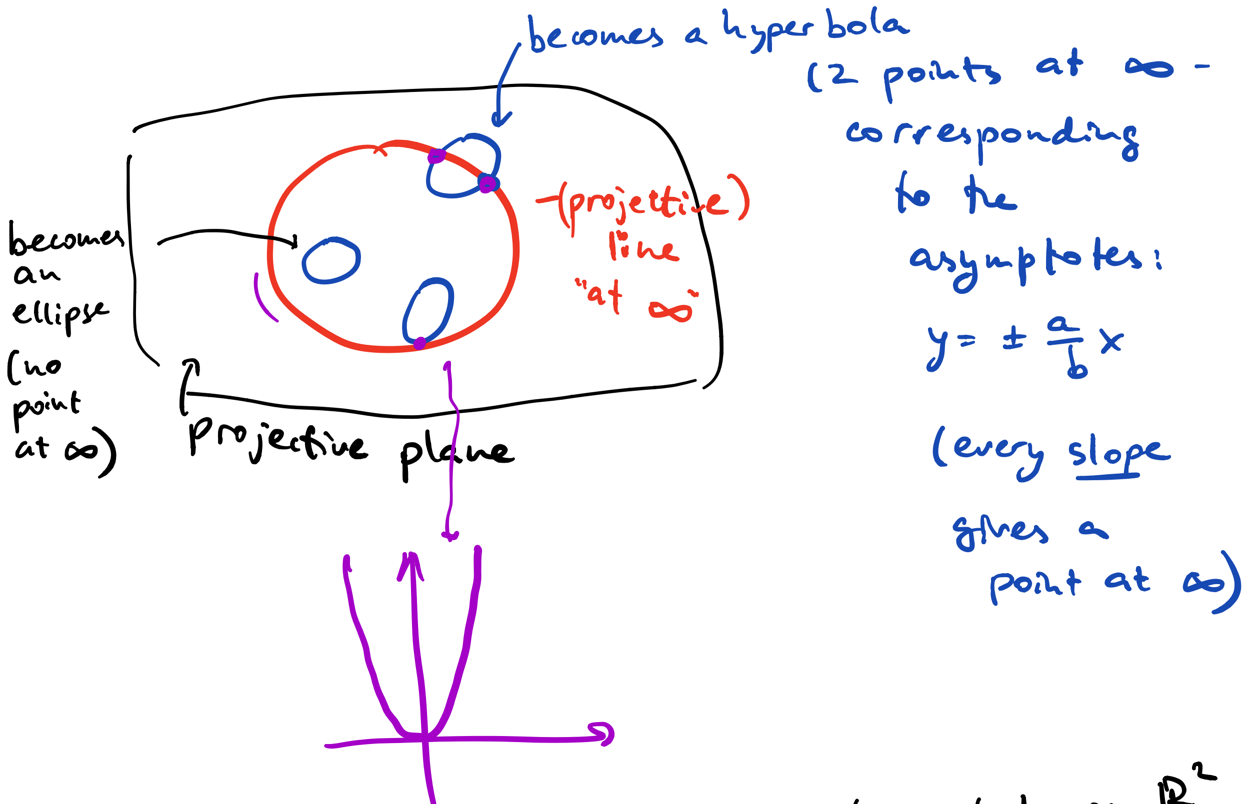
$$\boxed{a^2x^2 + y^2b^2 = c^2z^2}$$

$a, b, c \in \mathbb{R}$ (coefficients)

make it projective (homogeneous)

can go back to the ellipse by saying $x' = \frac{x}{z}$

$y' = \frac{y}{z}$



Example: $x^2 - y^2 = 1$ defines a hyperbola on $\mathbb{R}^2$

Make it projective: $x^2 - y^2 = z^2$.

Now it becomes $x^2 = y^2 + z^2$.

If we go to the new coordinates $(x:y:z) \rightsquigarrow (\frac{y}{x}, \frac{z}{x})$

it becomes an ellipse!

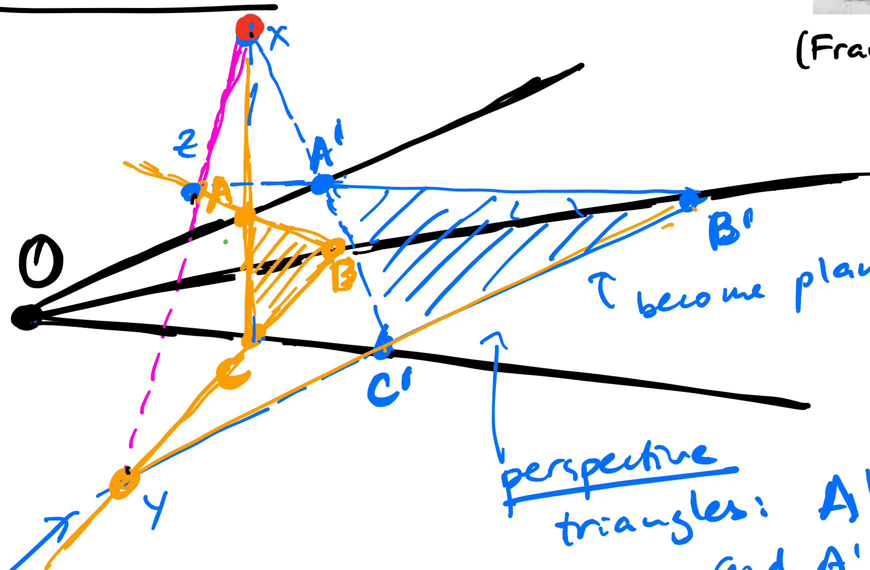
The cool stuff you can do with it

Girard Desargues

(France, 1591–1661)

Desargues' Theorem

Start with 3 lines passing through a point



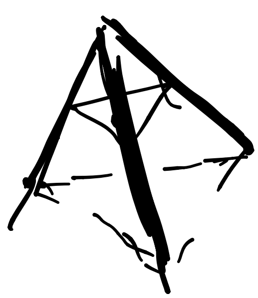
↑ become planes.

perspective triangles: ABC and A'B'C'

Then X, Y, Z lie on the same line.

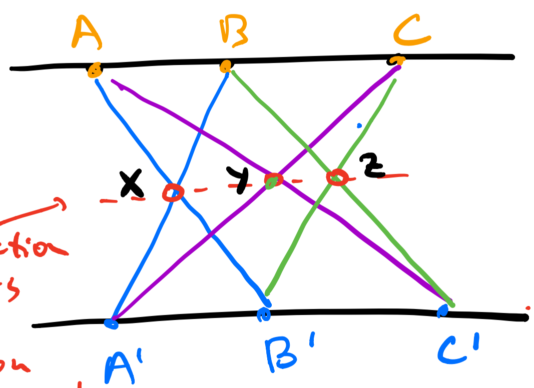
all our points lie on the line of intersection of the blue plane and the orange plane.

Pf: imagine that this is a picture in $\mathbb{R}^3$.



Note: you could use X or Y or Z as "O"

Pappus' Theorem



intersection points wind up on the same line, parallel to the two lines we started with.

Pappus of Alexandria
(~290 - 350 AD)

← parallel lines

Take A, B, C on one line, A', B', C' on the other.

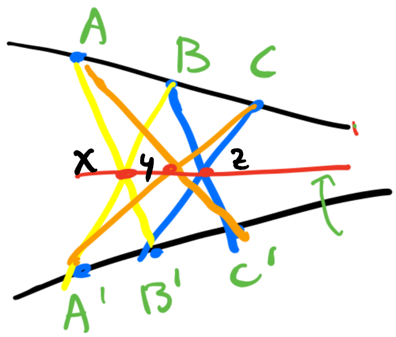
Make the lines AB, $A'B'$ → call their intersection point X

Let Y = intersection of AC' and $A'C$

Z = BC' and $B'C$.

Then X, Y, Z are on the same line parallel to the original two lines.

There is also a similar statement if you start with a pair of lines that intersect:



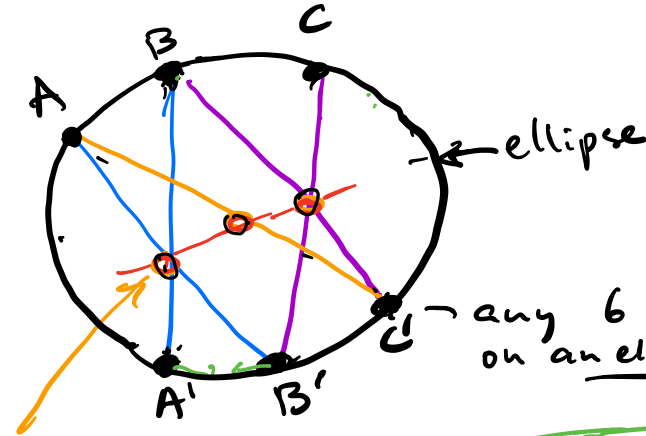
← if these lines intersect, the red line passes through the intersection point.

Main Point: any statement about points lying on the same line are projective

(so you can think of this theorem as a theorem on the projective plane, and that makes both cases: parallel lines or intersecting lines — into the same statement).

Note: Next is Pascal's Theorem, where instead of two lines we start with an ellipse.

Pascal's Hexagrammum Mysticum



any 6 points on an ellipse

lie on the same line!

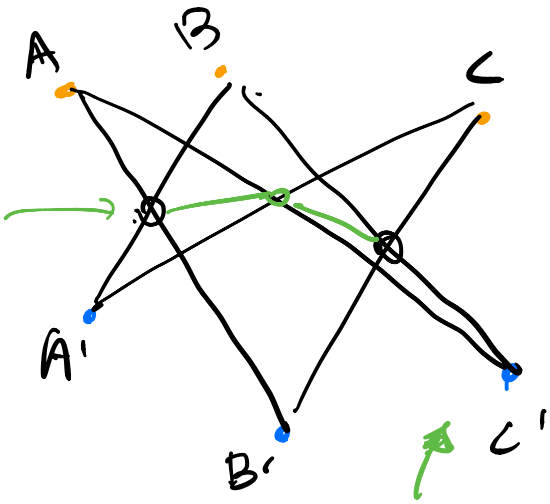
(Blaise Pascal, France, 1623–1662)

5 points define an ellipse
(so not any 6 points lie in an ellipse)

Pappus' Theorem is a special case of Pascal's Theorem!

Note: this is not true for random 6 points:

random 6 points.



not a line

They lie on the same degree 2 curve ('conic') iff Pascal's Thm holds

my 6 pts are not on any ellipse.

Note: two lines (parallel or intersecting) are a special case of a conic: suppose the lines in projective $(x:y:z)$ coordinates are given by: $a_1x+b_1y+c_1z=0$ and $a_2x+b_2y+c_2z=0$

Then their union is given by:

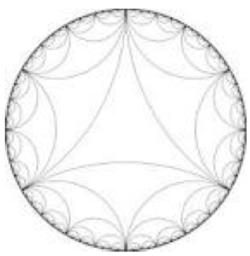
$$(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)=0,$$

which is a degree 2 equation!

(This is how Pappus' theorem becomes a special case of Pascal's Theorem)

Now what if instead of <u>NO</u> parallel lines, we require <u>two</u> 'parallel' lines through a given point?

Lobachewski, Minkowski, Einstein ...



yperbolic Geometry

?!?!

How dare you?!