RECURRENCE RELATIONS

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1. Homogeneous Linear Recurrence Relations

A homogeneous linear recurrence relation has the form

$$f_{n+1} = a_0 f_n + a_1 f_{n-1} + \cdots + a_k f_{n-k},$$

where $a_0, \ldots, a_k$ are constants. The aim is to find a closed-form formula for f_n.

Problem 1. Consider the relation $a_{n+1} = 2a_n$, $a_0 = 1$. What is a_n?

Problem 2 (The Fibonacci Sequence)**.** The Fibonacci sequence is given by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+1} = f_n + f_{n-1}, \forall n \geq 1.$$

What is f_{10}? How about f_{2020}? Find a closed-form formula for f_n.

Problem 3. Let $a_{n+1} = 5a_n - 6a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 4. Let $a_{n+1} = 4a_n - 4a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 5. Let $a_{n+1} = 2a_n - 2a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 6. Let $a_{n+1} = 4a_n - a_{n-1} - 6a_{n-2}$, $a_0 = 1, a_1 = 2, a_2 = 3$. Find a closed-form formula for a_n.

Problem 7 (The Gambler's Ruin Problem)**.** Smith has \$$n$ at the beginning of the day, and starts playing the following gambling game. At each step he tosses a coin, which comes up Heads with probability $\frac{1}{2}$, and Tails with probability $\frac{1}{2}$. If the coin comes up Heads, Smith gains \$1, and if it comes up Tails, he loses \$1. The game ends if either Smith has a total of \$$N$, where $N > n$, or if he has no money left. Find the probability q_n of Smith winning (i.e. having \$$N$) if he starts the day with \$$n$.

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2. Non-homogeneous Linear Recurrence Relations

A non-homogeneous linear recurrence relation has the form

$$f_{n+1} = a_0 f_n + a_1 f_{n-1} + \cdots + a_k f_{n-k} + g(n),$$

where $a_0, \ldots, a_k$ are constants, and $g(n)$ is a function that depends on n. The aim, again, is to find a closed-form formula for the n-th term f_n.

The general algorithm for solving such a relation is to first find a *particular solution*, x_n. Then, the sequence $(f_n - x_n)$ satisfies the homogeneous recurrence relation:

$$(f_{n+1} - x_{n+1}) = a_0(f_n - x_n) + a_1(f_{n-1} - x_{n-1}) + \cdots + a_k(f_{n-k} - x_{n-k}),$$

and, therefore, we can solve it using the tools we learned above.

Problem 8. Solve the recurrence relation

$$a_{n+1} = 3a_n + 1, \quad a_0 = 0.$$

Problem 9. Find all solutions to the recurrence relation

$$a_{n+1} = 3a_n + 4a_{n-1} + 3.$$

Problem 10 (The Towers of Hanoi)**.** Suppose we have 3 pegs, and there are n disks of increasing size on one of the pegs. The goal is to move all n disks to one of the other 2 pegs. We are only allowed to move one disk at a time, and cannot put a larger disk on top of a smaller one. Let H_n be the number of moves it takes to move the n disks. Show that H_n satisfies the recurrence relation

$$H_n = 2H_{n-1} + 1, \quad H_0 = 0,$$

and then solve this relation.

Problem 11 (The Binary Search Algorithm)**.** Suppose we are given n ordered real numbers $a_1 < a_2 < \cdots < a_n$, and another real number b. How many times do we have to check whether $b_j < a_j$ for some j in order to find the unique $i \in \{0, 1, \ldots, n\}$ so that $a_i \leq b < a_{i+1}$? *It might be easier to assume that n is a power of 2.*

Problem 12. Find all solutions to the recurrence relation

$$a_{n+1} = 2a_n + n, \quad a_0 = 0.$$

Now, try solving all of the problems above using the method of generating functions!