UBC Math Circle 2022 Problem Set 1

1. Prove that the Diophantine equation
$$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x + xyz = 0$$
has no solutions in nonzero integers. (Hint: Consider the parity of the left hand side in various cases.)

2. Let k be a positive integer. The sequence $(a_n)_n$ is defined by $a_1 = 1$, and for $n \geq 2$, a_n is the nth positive integer greater than a_{n-1} that is congruent to n modulo k. Find a_n in closed form.

3. (a) Show that there exist infinitely many integers x, y and z such that
$$x^2 + y^2 = 2z^3 + 8.$$
   (b) Show that there exist infinitely many integers a, b, c such that
$$a^2 + b^2 = c^2 + 3.$$

4. A subset S of $\mathbb{N}$ is called *highly composite* if for every $n \geq 2$ and every choice of distinct elements $a_1, a_2, \ldots, a_n \in S$, the sum $\sum_{i=1}^{n} a_i$ is composite. For example, the set $\{3, 5, 7\}$ is highly composite since $3 + 5$, $3 + 7$, $5 + 7$ and $3 + 5 + 7$ are all composite.
   (a) Prove or disprove: There exists an infinite highly composite set S containing only prime numbers.
   (b) Can the set P of all primes be partitioned into infinite highly composite subsets?

5. Given a positive integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest solution of equation
$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor$$
that exceeds a_{n-1}. Prove that all primes are among the terms of the sequence $a_1, a_2, \ldots$