UBC Math Circle 2022 Problem Set 2

1. Let
$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$$
be an identity in x. Find $a_0 + a_2 + a_4 + \cdots + a_{2n}$ in terms of n.

2. Let $f(x) = x^2 - 2$. For each $n \in \mathbb{N}$, we let $f^{\circ n} = f \circ f \circ \ldots \circ f$ (n times). Prove that for each $n \in \mathbb{N}$ there exist 2^n real numbers x such that $f^{\circ n}(x) = x$. (Hint: Let x be a real number such that $f^{\circ n}(x) = 0$ or ± 2 and consider what happens to x under $f^{\circ(n+1)}$.)

3. Let S be a subset of $\mathbb{R}^2$. It is called *convex* if for (a,b), $(c,d) \in S$, the line segment joining (a,b) and (c,d) lies entirely in S. It is *centrally symmetric* if whenever $(a,b) \in S$, then $(-a,-b) \in S$. Prove that if the area of a convex and centrally symmetric set S is greater than 4, then S contains a point of $\mathbb{Z}^2$ other than $(0,0)$. (Hint: Consider the map $(x,y) \mapsto (x \bmod 2, y \bmod 2)$ on S. Can this map be injective if the area of S is greater than 4?)

4. (a) Suppose $\alpha \in \mathbb{C}$ is a root of some nonzero polynomial in $\mathbb{Q}[x]$. Write $\mathbb{Q}[\alpha]$ to denote the set $\{P(\alpha) \mid P \in \mathbb{Q}[x]\}$. Show that for any $\beta \in \mathbb{Q}[\alpha] \setminus \{0\}$, there exists $\gamma \in \mathbb{Q}[\alpha]$ such that $\beta\gamma = 1$. (You may assume that $\mathbb{Q}[\alpha]$ is a finite-dimensional $\mathbb{Q}$-vector space, a consequence of which is that there exists $n \in \mathbb{N}$ such that for all $v_1, \ldots, v_n \in \mathbb{Q}[\alpha]$, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ not all zero such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$.)

   (b) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$. You may assume that α and $1 + \alpha + \alpha^2$ are nonzero. Write α^{-1} and $(1+\alpha+\alpha^2)^{-1}$ as elements of $\mathbb{Q}[\alpha]$.

5. Let $P(z) = a_d z^d + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients. The *reverse* of P is defined by
$$P^*(z) = \overline{a_0} z^d + \overline{a_1} z^{d-1} + \cdots + \overline{a_d}.$$

   (a) Prove that
$$P^*(z) = z^d \overline{P\left(\frac{1}{\overline{z}}\right)}.$$

   (b) Let m be a positive integer and let $q(z)$ be a monic nonconstant polynomial with complex coefficients. Suppose that all roots of $q(z)$ lie inside or on the unit circle. Prove that all roots of the polynomial
$$Q(z) = z^m q(z) + q^*(z)$$
lie on the unit circle.