UBC Math Circle 2022 Problem Set 3

1. Let $g : \mathbb{C} \to \mathbb{C}, \omega \in \mathbb{C}, a \in \mathbb{C}, \omega^3 = 1$, and $\omega \neq 1$. Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), z \in \mathbb{C},$$

and find the function f.

Solution: (Joanna Weng)

Since $\omega^3 = 1$ and $\omega \neq 1$, it follows that $\omega^2 + \omega + 1 = \frac{\omega^3 - 1}{\omega - 1} = 0$.

Then substituting $\omega z + a$ for z in a cyclic manner we obtain the following equations.

$$\begin{aligned} g(z) &= f(z) + f(\omega z + a). & (1) \\ g(\omega z + a) &= f(\omega z + a) + f(\omega^2 z + \omega a + a). & (2) \\ g(\omega^2 z + \omega a + a) &= f(\omega^2 z + \omega a + a) + f(\omega^3 z + \omega^2 a + \omega a + a) \\ &= f(\omega^2 z + \omega a + a) + f(z). & (3) \end{aligned}$$

Adding (1), (2), and (3), we have

$$f(z) + f(\omega z + a) + f(\omega^2 z + \omega a + a) = \frac{1}{2}\left[g(z) + g(\omega z + a) + g(\omega^2 z + \omega a + a)\right].$$

Then subtracting (2) from this gives

$$f(z) = \frac{1}{2}\left[g(z) - g(\omega z + a) + g(\omega^2 z + \omega a + a)\right].$$

2. Let f satisfy the functional equation

$$f(x)^2 = 1 + xf(x+1)$$

and the inequalities

$$\frac{x+1}{2} \leq f(x) \leq 2(x+1)$$

for all $x \geq 1$. Prove that $f(x) = x + 1$.

Solution: (Young Lin)

Substituting $x + 1$ for x in the given inequalities we obtain

$$\frac{x+2}{2} \leq f(x) \leq 2(x+2). \quad (1)$$

Solutions edited by Josh Gomes and Victor Wang.

Now given that $f(x)^2 = 1 + xf(x+1)$, it follows that

$$\frac{1}{2} + xf(x+1) < f(x)^2 < 2 + xf(x+1). \tag{2}$$

Putting (1) and (2) together, we obtain

$$\frac{(x+1)^2}{2} = \frac{1}{2} + x\left(\frac{x+2}{2}\right) < f(x)^2 < 2 + x(2(x+2)) = 2(x+1)^2,$$

which by taking square roots gives

$$2^{-1/2}(x+1) < f(x) < 2^{1/2}(x+1).$$

Observe that the bound on f has improved. So applying the same procedure n times we obtain

$$2^{-1/2^n}(x+1) < f(x) < 2^{1/2^n}(x+1),$$

which in the limit as $n \to \infty$ gives

$$x + 1 \leq f(x) \leq x + 1.$$

Hence, $f(x) = x + 1$ as desired.

3. Let $f(x) = x^2 + 2022x + 1$. Define $f^{\circ n} = f \circ f \circ \ldots \circ f$ (n times). Prove that $f^{\circ n}$ has at least two real roots.

Solution: (Oakley Edens)

It is clear that $f^{\circ n}(1) > 0$ and $f^{\circ n}(-2022) > 0$ for all n. Next, let α be the unique negative root of $f(x) - x = x^2 + 2021x + 1 = 0$, obtained from the quadratic formula. Then $f(\alpha) = \alpha$ and it follows that $f^{\circ n}(\alpha) = \alpha < 0$. By the intermediate value theorem, $f^{\circ n}$ has a root in the intervals $(-2022, \alpha)$ and $(\alpha, 1)$. Thus $f^{\circ n}$ has at least two real roots.

4. For every $n \geq 0$, find all polynomials $f(x) \in \mathbb{Z}[x]$ such that for all $x \in \mathbb{C} \setminus \{0\}$,

$$f\left(x + \frac{1}{x}\right) = x^n + \frac{1}{x^n}.$$

Your solution may be written in the form of a recurrence.

Solution: (Oakley Edens)

Suppose that for a given n, there exists a polynomial $f_n(x)$ satisfying the given equality. Define $g_n(x) = \frac{1}{2}f_n(2x)$.

Note that $g_n(x)$ satisfies the functional equation

$$g_n\left(\frac{x + x^{-1}}{2}\right) = \frac{x^n + x^{-n}}{2}.$$

Next, we substitute $x = e^{i\theta}$. Using Euler's Identities, the equation becomes

$$g_n(\cos\theta) = \cos(n\theta).$$

Since $\cos(n\theta)$ is a polynomial in $\cos\theta$, we can guarantee that a unique solution g_n will exist for all n. This implies that a unique solution f_n exists for all n. To describe these solutions we begin by using angle sum formulas to obtain

$$\cos(n\theta) = \cos((n-1)\theta)\cos\theta - \sin((n-1)\theta)\sin(\theta)$$

and

$$\cos((n-2)\theta) = \cos((n-1)\theta)\cos\theta + \sin((n-1)\theta)\sin\theta.$$

Adding these two equalities gives

$$\cos(n\theta) = 2x\cos((n-1)\theta)\cos\theta - \cos((n-2)\theta).$$

Using the functional equation for g_n this implies that g_n satisfies the recurrence

$$g_n(x) = 2xg_{n-1}(x) - g_{n-2}(x)$$

with $g_0(x) = 1$ and $g_1(x) = x$. Rewriting this in terms of $f_n(x)$ (and after some manipulation) we obtain

$$f_n(x) = xf_{n-1}(x) - f_{n-2}(x)$$

with $f_0(x) = 2$ and $f_1(x) = x$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y$$

for all $x, y \in \mathbb{R}$.

Solution:

Begin by observing that $f(x) = \pm x$ are solutions. Next, substituting 0 for x and y, we see that

$$f(f(0)^2 + f(0)) = 0.$$

Hence, there is some $u \in \mathbb{R}$ such that $f(u) = 0$. Now substituting $x = u$, we find that

$$f(f(y)) = y \text{ for all } y \in \mathbb{R}.$$

So f is an involution and thus a bijection.

Since f is surjective, for each $x \in \mathbb{R}$ there is a $t \in \mathbb{R}$ such that $f(t) = x$. So substituting $f(t)$ for x we get

$$f(f(f(t))^2 + f(y)) = f(t)f(f(t)) + y \implies f(t^2 + f(y)) = tf(t) + y.$$

But $f(f(t)^2 + f(y)) = tf(t) + y$ by the given equation, and since f is injective, it follows that

$$f(t)^2 + f(y) = t^2 + f(y) \implies f(t) = \pm t.$$

Finally, we verify that either $f(t) = t$ for all $t \in \mathbb{R}$ or $f(t) = -t$ for all $t \in \mathbb{R}$. Suppose $f(a) = a$ and $f(b) = -b$ for $a, b \in \mathbb{R}$. It is enough to show that one of a or b is zero. Observe that

$$f(a^2 + f(y)) = a^2 + y \text{ and } f(b^2 + f(y)) = -b^2 + y \text{ for all } y \in \mathbb{R}.$$

But $f(0) = 0$, which means that $f(a^2) = a^2$ and $f(b^2) = -b^2$. Then

$$f(a^2 - b^2) = f(a^2 + f(b^2)) = a^2 - b^2 = f(b^2 + f(a^2)) = f(b^2 + a^2),$$

and since f is injective we have that

$$a^2 - b^2 = a^2 + b^2 \implies b = 0.$$

Hence, $f(x) = \pm x$ are the only solutions.