UBC Math Circle 2022 Problem Set 4

1. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum \alpha_i = n$. Prove that

   (a) The number of compositions of n is 2^{n-1}.

   (b) The total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

   (c) For $n \geq 2$, the number of compositions of n with an even number of even parts is equal to 2^{n-2}.

Solution: (Victor Wang)

(a) Note there is a bijection between the compositions of n and subsets of $[n-1] = \{1, \ldots, n-1\}$, sending $(\alpha_1, \ldots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. Since there are 2^{n-1} subsets of $[n-1]$, it follows that there are 2^{n-1} compositions of n.

(b) The number of parts in a composition $(\alpha_1, \ldots, \alpha_k)$ of n exactly one more than the number of elements in the corresponding subset $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$ of $[n-1]$. Note the bijection from the subsets of $[n-1]$ to itself sending $S \mapsto [n-1] \setminus S$ satisfies $|S| + |[n-1] \setminus S| = n-1$. Hence the total number of parts of all compositions of n is equal to

$$\frac{1}{2} \sum_{S \subseteq [n]} (|S| + 1 + |[n-1] \setminus S| + 1) = \frac{1}{2}(n+1)2^{n-1} = (n+1)2^{n-2}.$$

(c) Consider the bijection from the set of compositions of n to itself sending $(\alpha_1, \ldots, \alpha_k) \mapsto (1, \alpha_1 - 1, \ldots, \alpha_k)$ if $\alpha_1 > 1$, and $\mapsto (\alpha_1 + \alpha_2, \ldots, \alpha_k)$ if $\alpha_1 = 1$. (What does this involution correspond to under the bijection to subsets of $[n-1]$?) This bijection changes the parity of the count of even parts. Hence, exactly half of all compositions of n have an even number of even parts, so the number of compositions of n with an even number of even parts is 2^{n-2}.

2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Solutions edited by Josh Gomes and Victor Wang.

Solution: (Joanna Weng)

Consider the $2^{N-2} \geq N \geq 4$ distinct flags with yellow in the first column and blue in the second column. Among these flags, it is impossible to arrange N of them into an $N \times N$ square so that all fields on its main diagonals will have the same colour. Hence, $M > 2^{N-2}$.

We show that $M = 2^{N-2} + 1$. To do this we make use of the following theorem.

Theorem 1 (Hall's Marriage Theorem)**.** *Suppose G is a bipartite graph with bipartition (A, B). There is a matching that covers A if and only if for every $X \subseteq A$, $N_G(X) \geq |X|$ where $N_G(X)$ is the number of neighbours of X.*

Suppose for a contradiction that $M > 2^{N-2} + 1$. Then there exists some set of flags F of size $2^{N-2}+1$ that does not have an N-subset (a subset of size N) that is diverse. Let C be the set of all columns, i.e., the set $\{1, 2, \ldots, N\}$. We define bipartite graphs G and G' with bipartition (C, F) so that a flag $f \in F$ is incident to a column $c \in C$ if and only if f has colour w at column c, where w is yellow for G and blue for G'. It is easy to see that any matching of G or G' that covers C produces an N-subset of F that is diverse. So G and G' cannot have a matching that covers C. By Theorem 1, this means that there are nonempty subsets X and Y of C such that $N_G(X) < |X|$ and $N_{G'}(Y) < |Y|$. This implies that the number of flags in F that have a blue field at a column in X is at most $|X| - 1$, and the number of flags in F that have a yellow field at a column in Y is at most $|Y| - 1$. So the maximum number of flags in F that do not have a blue field at a column in X or a yellow field at a column in Y is $2^{N-|X|-|Y|}$. It then follows that

$$2^{N-2}+1 \leq (|X|-1)+(|Y|-1)+2^{N-|X|-|Y|} \implies 2^{N-2}+3 \leq |X|+|Y|+2^{N-|X|-|Y|}.$$

If $|X| + |Y| \geq 2$, the above inequality will of course be false. So $|X| + |Y| \leq 1$. But this is also not possible for then either $|X|$ or $|Y|$ is zero, making X or Y empty. So it must be the case that $M = 2^{N-2} + 1$ as desired.

3. N cells are chosen on a rectangular grid. Let a_i is number of chosen cells in i-th row, b_j is number of chosen cells in j-th column. Prove that

$$\prod_i a_i! \cdot \prod_j b_j! \leq N!$$

Solution: (Arvin Sahami)

Let the grid have m rows and n columns. Assume that we want to count the number of permutations of $1, 2, \ldots, n$ over the N chosen cells. It is easy to check that this number is $N!$.

In what follows, we define our permutation function over the N marked cells only, and we ignore the unmarked cells.

Call a permutation function r over the N cells row-wise if it keeps all cells in their row, i.e., when given a permutation $(x_1, x_2, \ldots, x_N)$ as input and $r(x_1, x_2, \ldots, x_N) = (y_1, y_2, \ldots, y_N)$ as output, then x_i and y_i are on the same row. We can define a column-wise permutation similarly.

Now note that there are at most

$$A = \Pi_1^m (a_i)!$$

row-wise permutations for any row, and there are $a_i!$ ways to permute the marked cells of row i. We should multiply all of these numbers to get all possible permutations. Let $\{r_1, r_2, \ldots, r_A\}$ be the set of these permutations.

In the same fashion, we can show that there are at most

$$B = \Pi_1^n (b_i)!$$

column-wise permutations. Let the set of these permutations be $\{c_1, c_2, \ldots, c_B\}$.

Now, considering $I = (1, 2, \ldots, N)$ as out initial permutation, we will look at permutations of the form $r_i(c_j(I))$ for $1 \leq i \leq m, 1 \leq j \leq n$.

First, we show the following:

Claim: The two permutations

$$r_i(c_j(I)), r_k(c_\ell(I))$$

are equal if and only if $i = k$ and $\ell = j$.

Proof. First, note that every permutation has an inverse. Now, observe that the inverse permutation of a row-wise (resp. column-wise) permutation must be row-wise (resp. column-wise) and combination of two row-wise (resp. column-wise) permutations is row-wise (resp. column-wise). Letting r_i^{-1} to be the inverse of r_i, we have:

$$r_i(c_j(I)) = r_k(c_\ell(I)) \implies r_i^{-1}(r_i(c_j(I))) = r_i^{-1}(r_k(c_\ell(I))).$$

Now $r_i^{-1}(r_k(x))$ is also a row-wise permutation, say r_z. This implies that

$$c_j(I) = r_z(c_\ell(I)).$$

But this means that r_z should be the identity permutation since if r_z actually moves something through a row, c_j on the LHS will fail to do so! Hence,

$$r_z = \text{id} \implies c_j(I) = c_\ell(I) \implies j = \ell$$

and

$$r_z = r_i^{-1} r_k = \text{id} \implies r_i \cdot \text{id} = r_i = r_k \implies i = k$$

as desired.

Finally, getting back to the problem, let S be the set of permutations reachable by $r_i(c_j(I))$. For different values of i, j, the above claim shows that $|S| = A \cdot B$ since we have a unique permutation for each choice of A, B.

It is obvious that $|S| \leq N!$ since S is a subset of the set of all permutations. Hence, the inequality

$$\Pi_{i=1}^{m} a_i! \cdot \Pi_{i=1}^{n} b_i! = A \cdot B = |S| \leq N!$$

follows as desired!

4. You are given an unbiased fair coin C. Can you use C to simulate a biased coin C' which produces heads with probability p such that on average C is flipped twice? (i.e. either come up with a procedure which simulates C' and flips C twice on average or prove that no such procedure exists).

Solution: (Oakley Edens)

Let $p = 0.p_1p_2\ldots$ be the binary representation of p. Next, we flip C, recording a binary number $q = 0.q_1q_2\ldots q_n$ (with q_i the result of the i-th coin flip: 0 if heads, 1 if tails) until that value of n where it becomes clear that $q \leq p$ or $q > p$. If $q \leq p$ then the result of the simulated coin flip is heads while if $q > p$ the result is tails. This procedure produces a random number in the interval $[0, 1]$. Thus it has probability p of being chosen in the interval $[0, p]$. This proves that the procedure accurately simulates a biased coin of probability p. Finally, the simulation ends after k steps if and only if $q_i = p_i$ for all $1 \leq i \leq k-1$ but $q_k \neq p_k$. This has probability $\frac{1}{2^k}$. The expected number of flips is then $\sum_{i=1}^{\infty} \frac{i}{2^i}$. To evaluate this sum, we take the geometric series formula $\sum_{i=1}^{\infty} x^i = \frac{1}{1-x}$ for $|x| < 1$ and differentiate both sides. This gives $\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$. Multiplying by x gives that $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$. Observe that $x = \frac{1}{2}$ is well within the radius of convergence, thus we get an average of $\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$ flips.

5. The Fibonacci numbers may be defined by the recurrence

$$F_0 = 0,\ F_1 = 1,$$

and

$$F_n = F_{n-1} + F_{n-2}$$

for $n > 1$.

Show that

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

for all $n \geq 1$ and $m \geq 0$. (Possible solution hint: observe that for $n \geq 1$, the number of possible ways of tiling a $1 \times (n-1)$ rectangle with monominos and dominos is equal to F_n.)

Solution: Following the hint, we first note that F_n for $n \geq 0$ is the number of ways $a(n-1)$ of tiling a $1 \times (n-1)$ rectangle with monominos and dominos. This is because $a(-1) = 0$ and $a(0) = 1$ (both in a vacuous sense), and for larger n when tiling a $1 \times (n-1)$ rectangle, either the first square is filled with a monomino and there are $a(n-2)$ ways to complete the tiling, or the first two squares are covered by a domino and there are $a(n-3)$ ways to complete the tiling (so $a(n-1) = a(n-2) + a(n-3)$).

When tiling a $1 \times (n+m-1)$ rectangle, either the $(n-1)$st and nth square are covered by the same domino in which case there are $F_{n-1}F_m$ ways to complete the tiling, or the tiling decomposes into a tiling of the first $n-1$ squares and the last m squares in which case there are F_nF_{m+1} ways to do so. Hence $F_{n+m} = F_{n-1}F_m = F_nF_{m+1}$, as desired.