UBC Math Circle 2022 Problem Set 5

1. Find all solutions to the equation $E + V + F = G + 2$ where E, V, F, G are positive integers and E, V, F all divide G.

Solution: Since E, V, F all divide G, we can write $p = G/E$, $q = G/V$, $r = G/F$ and study the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{2}{G} + 1,$$

where p, q, r are positive integers.

Suppose without loss of generality that $p \le q \le r$. Then $p \le 2$ since otherwise

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1 < \frac{2}{G} + 1.$$

Case 1: Suppose $p = 1$. This forces $q = G = r$.

Case 2: Suppose $p = 2$. Then $\frac{1}{q} + \frac{1}{r} = \frac{2}{G} + \frac{1}{2}$. It must be that $q \le 3$ for otherwise

$$\frac{1}{q} + \frac{1}{r} \le \frac{1}{2} < \frac{2}{G} + \frac{1}{2}.$$

Subcase 1: Suppose $q = 2$. Then $r = G/2$.

Subcase 2: Suppose $q = 3$. Then $G = \frac{12r}{6-r}$. Since $r \ge 3$, r can be 3, 4, or 5.

Putting these together we see that

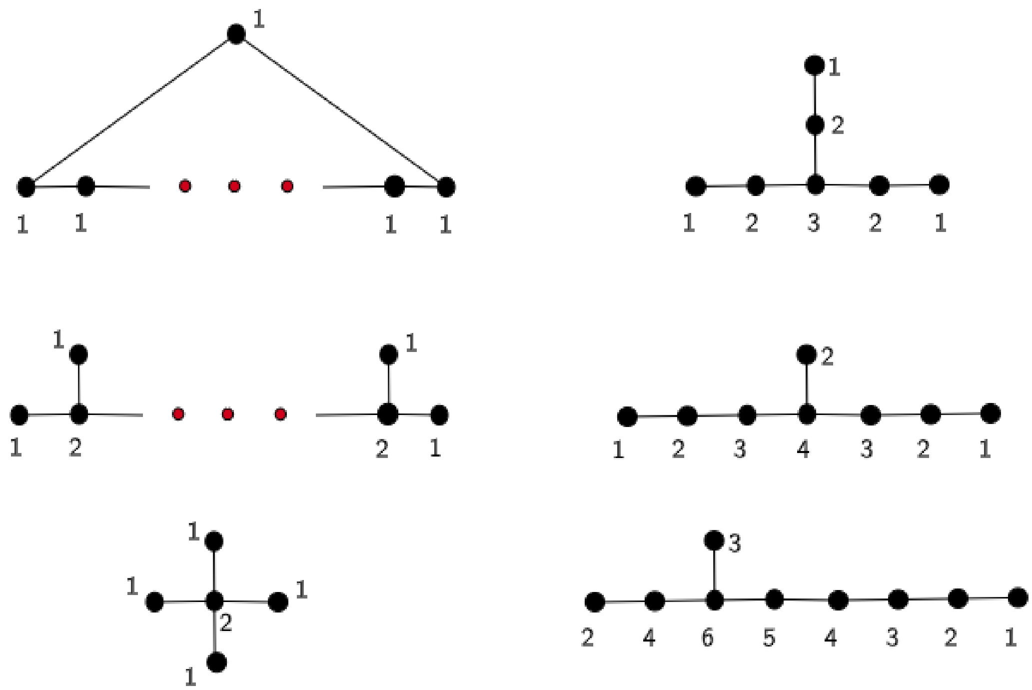
$$\begin{aligned} 1 + 1 + n &= 2 + n && \text{(natural } n\text{)} \\ n + n + 2 &= 2 + 2n && \text{(natural } n\text{)} \\ 4 + 4 + 6 &= 2 + 12 \\ 6 + 8 + 12 &= 2 + 24 \\ 12 + 20 + 30 &= 2 + 60 \end{aligned}$$

are solutions. In the ADE classification, the above solutions are labelled by A_{n-1}, D_{n+2}, and E_6, E_7, E_8.

2. A population on a graph is an assignment of positive integers to each vertex. A *perfect population* has the property that the population of each vertex is exactly $1/2$ of the sum of the neighbouring populations. Find all perfectly populated (finite) graphs.

Solution: Below are the six possible "primitive" (connected nontrivial) perfectly populated graphs.

Solutions edited by Josh Gomes and Victor Wang.



Below is an argument to recover the above solutions.

Let G be a (connected nontrivial) perfectly populated graph. We leave it as an exercise to show that:

(i) If G is not a tree, then G is a cycle.

(ii) G cannot be a path.

(iii) $\deg(v) \leq 4$ for all vertices v of G.

(iv) If G contains a vertex of degree 4, then G is the star graph S_4.

From (i) the only possible perfectly populated graph that is not a tree is a cycle, which gives our first solution. From (iv) we can verify that S_4 is indeed a solution, which gives our second solution. And so by (i)-(iii) the only solutions remaining are trees that must have a vertex of degree 3 and no larger.

Suppose G has as a vertex v of degree 3 with integer label a. Let $a - d_1$, $a - d_2$, and $a - d_3$ be the integer labels of the neighbours of v where d_1, d_2, d_3 are nonnegative. It is not hard to verify that the labels of the vertices in the paths leading to v form a (finite) arithmetic progression with differences d_1, d_2, d_3, respectively. So if G has 2 degree 3 vertices, then the labels of the vertices in the path joining them must be constant. Since it is possible for only one of d_1, d_2, d_3 to be zero, if G has 2 degree 3 vertices then G has exactly 2 degree 3 vertices. Hence, we can recover the only solution with 2 degree 3 vertices.

All remaining solutions must now have exactly 1 degree 3 vertex, which means d_1, d_2, d_3 are nonzero. Now if d_i is nonzero, then d_i divides a for $i = 1, 2, 3$. This is because each of the three paths leading away from v ends at a leaf. Then since $d_1 + d_2 + d_3 = a$, we can write $p = a/d_1$, $q = a/d_2$, $r = a/d_3$ and study the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

where p, q, r are positive integers. Suppose without loss of generality that $p \leq q \leq r$. Then it is not hard to see that $2 \leq p \leq 3$.

Case 1: Suppose $p = 2$. Then either $q = 4 = r$, or $q = 3$ and $r = 6$. This gives solutions $d_1 = a/2$, $d_2 = a/4$, $d_3 = a/4$, or $d_1 = a/2$, $d_2 = a/3$, $d_3 = a/6$.

Case 2: Suppose $p = 3$. Then $q = 3 = r$. This gives the solution $d_1 = a/3$, $d_2 = a/3$, $d_3 = a/3$.

We see that the above cases give us the remaining three solutions.

3. Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to colour the computers and cables such that no two computers have the same colour, no two cables joined to a common computer have the same colour, and no computer is assigned the same colour as any cable joined to it. Prove that this can be done using n colours. What about when n is even?

Solution: Arrange the n computers into a regular n-gon. Note that the cables, drawn as straight lines, can be partitioned into n sets of parallel lines (and no two parallel lines connecting vertices of the regular n-gon will be connected to the same vertex). So we can assign one colour to each set of parallel lines. For each colour, the cables given that colour will pair together vertices, so since n is odd there is some vertex left over, which we will give the same colour. This completes the solutiion.

4. Two pyramids with common base $A_1A_2A_3A_4A_5A_6A_7$ and vertices B and C are given. The edges $BA_i, CA_i (i = 1, \ldots, 7)$, the diagonals of the common base and the segment BC are coloured in either red or blue. Prove that there exists a triangle whose sides are colored in one and the same color.

Solution: (Joanna Weng)

Without loss of generality, suppose segment BC is red. We consider three cases.

Case 1: At least three of the edges BA_i are red.
Let the three red edges be BA_r, BA_s, and BA_t. At least one of the sides of the triangle $A_rA_sA_t$, say A_rA_s, is a diagonal of the base and is colored. If A_rA_s is red, then $\triangle BA_rA_s$ is all red. Moreover, if CA_r or CA_s is red, then $\triangle BCA_r$ or $\triangle BCA_s$ is all red. Thus all of A_rA_s, CA_r, CA_s must be blue to avoid an all-red triangle; but this makes A_rA_sC an all-blue triangle. Thus we must always have a unicoloured triangle in this case.

Case 2: Exactly two of the edges BA_i are red.
Let the two edges be BA_r and BA_s, and consider two subcases. If A_rA_s is a diagonal, we can reason as Case I. Otherwise, suppose WLOG that BA_i is blue for $i \neq 1, 2$. Consider the three base vertices A_3, A_5, A_7. Since BA_3, BA_5, BA_7 are all blue, the diagonals A_3A_5, A_5A_7, A_7A_3 must all be red to avoid an all-blue triangle with B. This forces $\triangle A_3A_5A_7$ to be all red. Thus we must always have a unicoloured triangle in this case.

Case 3: Exactly one of the edges BA_i is red.
Suppose WLOG that BA_1 is red. Reasoning as in Case II, $\triangle A_3A_5A_7$ must be all red.

There is a unicolored triangle in every case.

5. For $k \in \mathbb{Z}_{\geq 0}$, a *proper k-colouring* of a graph G with vertex set V and edge set E is a map $\kappa : V \to \{1, \ldots, k\}$ so that for all edges $uv \in E$, $\kappa(u) \neq \kappa(v)$. Let $\chi_G(k)$ denote the number of proper k-colourings of G.

   (a) Show that $\chi_G(k)$ is a polynomial in k. χ_G is known as *the chromatic polynomial.*

   (b) What are the chromatic polynomials of the path and complete graphs on n vertices? (The path P_n has n vertices labelled $1, \ldots, n$ with an edge between each pair of vertices labelled i and $i+1$. The complete graph K_n has n vertices and an edge between every pair of distinct vertices.)

   (c) An *acyclic orientation* of G is an assignment of a direction to each edge of G so that there are no directed cycles (i.e. there is no way to go from a vertex to itself by following directed edges). Show that the number of acyclic orientations of G is given by $|\chi_G(-1)|$.

Solution: (Victor Wang)

(a) Note that $\chi_G(k)$ satisfies a *deletion-contraction* relation: for any edge e, $\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$, where $G - e$ is the graph with e removed, and G/e is the graph obtained by formally identifying the endpoints of e in $G - e$. This is because in a proper k-colouring of $G - e$, either the endpoints of e are given different colours, which corresponds bijectively to the proper k-colourings of G, or the endpoints of e are given the same colour, which corresponds bijectively to the proper k-colourings of G/e.

We proceed by induction on the number of edges. If G has n vertices and no edges, then the vertices may be coloured in any way, so $\chi_G(k) = k^n$ is a polynomial. Otherwise, there is some edge e, and $G - e, G/e$ have strictly fewer edges. So $\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$ is a polynomial.

(b) If we try colouring the vertices of the path in order, there are k possible colours for the first vertex, and $k-1$ possible colours for each successive vertex (as the colour chosen just has to be distinct from the colour of its previous neighbour). So $\chi_{P_n}(k) = k(k-1)^{n-1}$.

If we try colouring the vertices of the complete graph in any order, there are k possible colours for the first vertex, $k-1$ for the second, $k-2$ for the third, etc. (as the colour chosen for a vertex has to be distinct from all previously chosen colours). So $\chi_{K_n}(k) = k(k-1)\cdots(k-n+1)$.

(c) (This result appears in Richard Stanley's 1973 paper "Acyclic orientations of graphs".)

We will show by induction on the number of edges m of G that the number of acyclic orientations of a graph is $(-1)^n \chi_G(-1)$, where n is the number of vertices. For the base case when $m = 0$, this is clear since $\chi_G(k) = k^n$ where n is the number of vertices of G, and the number of acyclic orientations is just $(-1)^n \cdot (-1)^n = 1$ (the trivial orientation).

For the inductive step, note that the number of acyclic orientations of G is equal to the sum of the number of acyclic orientations of $G - e$ and G/e where e is an edge of G connecting vertices u and v. To see this, note acyclic orientations of $G - e$ with a directed path from u to v are in bijection with acyclic orientations of G with a directed path from u to v (not including e) and e oriented from u to v. Similarly, acyclic orientations of $G - e$ with a directed path from v to u are in bijection with acyclic orientations of G with a directed path from v to u (not including e) and e oriented from v to u. And for every acyclic orientation of G with no directed path from u to v and no directed path from v to u, there are two corresponding acyclic orientations of G, and one corresponding acyclic orientation of G/e.

So by the inductive hypothesis, the number of acyclic orientations of G is $(-1)^n \chi_{G-e}(-1) + (-1)^{n-1} \chi_{G/e}(-1) = (-1)^n \chi_G(-1)$.