UBC Math Circle 2022 Problem Set 6

1. (a) Prove that for any nonempty subsets A and B of $\mathbb{R}$ we have

$$|A+B| \geq |A|+|B|-1,$$

where $A+B := \{a+b \mid a \in A, b \in B\}$.

(b) Prove that for any prime p and nonempty subsets A and B of $\mathbb{Z}/p\mathbb{Z}$ we have

$$|A+B| \geq \min\{p,\ |A|+|B|-1\},$$

where $A+B := \{a+b \pmod{p} \mid a \in A, b \in B\}$.

(Hint: You may use the fact that Combinatorial Nullstellensatz holds over $\mathbb{Z}/p\mathbb{Z}$.)

Solution:

(a) Note $|A+B| = |A'+B'|$ where $A' = \{a - \max_{x\in A} x : a \in A\}$ and $B' = \{b - \min_{y\in B} y : b \in B\}$. Note $0 \in A', B'$ and every element of A' is nonpositive, while every element of B' is nonnegative. Hence $A' \cup B' \subset A'+B'$, where $|A' \cup B'| \geq |A'|+|B'|-1 = |A|+|B|-1$, since $A' \cap B' = \{0\}$.

(b) Let $A, B \subset \mathbb{Z}/p\mathbb{Z}$ be nonempty. We will show $|A+B| \geq \min\{p, |A|+|B|-1\}$.

When $|A|+|B| > p$, then for all $c \in \mathbb{Z}/p\mathbb{Z}$, the intersection of A and $\{c\}-B$ is nontrivial by the pigeonhole principle, so every $c \in \mathbb{Z}/p\mathbb{Z}$ is in the sumset $A+B$, which shows the result in that case.

It remains to show the case when $|A|+|B| \leq p$. Let $f(x,y) = \prod_{c\in A+B}(x+y-c)$, $g(x) = \prod_{a\in A}(x-a)$, and $h(y) = \prod_{b\in B}(y-b)$. Suppose, for a contradiction, that $\deg(f) = |A+B| \leq |A|+|B|-2$.

Since $f(x,y)$ vanishes on $A \times B$, by combinatorial Nullstellensatz we have $f(x,y) = k(x,y)g(x) + \ell(x,y)h(y)$ for some polynomials

$k(x,y), \ell(x,y) \in \mathbb{Z}/p\mathbb{Z}[x,y]$ satisfying $\deg(k) \leq |A+B| - \deg(g)$ and $\deg(\ell) \leq |A+B| - \deg(h)$.

Note the coefficient of $x^{|A|-1}y^{|A+B|-|A|+1}$ is $\binom{|A+B|}{|A|-1}$. Since p is prime and $|A+B| \leq |A|+|B|-2 < p$, the coefficient is nonzero.

Since the degree of $x^{|A|-1}y^{|A+B|-|A|+1}$ is $|A+B|$ and $\deg(k)+\deg(g) \leq |A+B|$, any occurrence of the monomial $x^{|A|-1}y^{|A+B|-|A|+1}$ in $k(x,y)g(x)$ can only arise from multiplying highest degree terms from each of $k(x,y)$ and $g(x)$. The unique highest degree term in $g(x)$ is $x^{|A|}$, which has degree in x greater than the degree in x of $x^{|A|-1}y^{|A+B|-|A|+1}$, so the coefficient of $x^{|A|-1}y^{|A+B|-|A|+1}$ in $k(x,y)g(x)$ is zero. Similarly, the coefficient of $x^{|A|-1}y^{|A+B|-|A|+1}$ in $\ell(x,y)h(y)$

is zero, since $y^{|B|}$ is the unique highest degree term in $h(y)$ and $|B| > |A+B| - |A| + 1$, by assumption.

But then this would imply that the coefficient of $x^{|A|-1}y^{|A+B|-|A|+1}$ in $f(x, y)$ is zero, a contradiction.

So in all cases we have $|A + B| \geq \min\{p, |A| + |B| - 1\}$.

2. Prove that for any partition of the positive integers into a finite number of parts, one of the parts contains three integers x, y, z with $x + y = z$.

   (Hint: It may be helpful to know Ramsey's theorem, which may be stated as follows. For any positive integer k, there is a positive integer n such that any gathering of n people contains either k mutual friends or k mutual strangers.)

Solution: Note that the given version of Ramsey's theorem can be used to prove a "multicoloured" analogue. Namely, for any positive integers k, l, there exists a positive integer n such that if all edges of the complete graph K_n are coloured one of l colours, then there exists k vertices such that all edges between them are the same colour. (This can be proven by induction on l, first applying the 2-colour version with k sufficiently large, then the version for $l - 1$ colours on the monochromatic K_k.)

Let l be the number of parts and $k = 3$, and n be so that the statement of the multicoloured Ramsey's theorem is satisfied. Consider the complete graph on vertices 1 to n, where the edge from $i < j$ is given the "colour" corresponding to the part where $j - i$ lives. Then, there is a monochromatic triangle, that is, there exists $1 \leq i < j < k \leq n$ such that $j-i, k-j, k-i$ all are in the same part of the partition. But $(j - i) + (k - j) = (k - i)$, so this proves the problem statement.

3. For a prime p and a given integer n let $\nu_p(n)$ denote the exponent of p in the prime factorisation of $n!$. Given $d \in \mathbb{N}$ and $\{p_1, p_2, \ldots, p_k\}$ a set of k primes, show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

Solution: (Neo Huang)

For a positive integer x, let $u_p(x)$ denote the exponent of p in the factorization of x. We now note two properties of ν_p:

$$\nu_p(x) = \sum_{r=1}^{x} u_p(r)$$

$$\nu_p(x + y) = \nu_p(x) + \nu_p(y) \text{ when } u_p(x) \neq u_p(y).$$

Since $u_p(nm) = u_p(n) + u_p(m)$ for any integers n and m, the first property follows because $\nu_p(x) = u_p(x!)$. For the second property, we have that $u_p(x+y) = \min\{u_p(x), u_p(y)\}$ if $u_p(x) \neq u_p(y)$. In particular, assuming without loss of generality that $u_p(x) > u_p(y)$,

$$\begin{aligned}\nu_p(x+y) &= \sum_{r=1}^{x+y} u_p(r) = \sum_{r=1}^{x} u_p(r) + \sum_{r=x+1}^{x+y} u_p(r) \\ &= \nu_p(x) + \sum_{r=1}^{y} u_p(r) = \nu_p(x) + \nu_p(y).\end{aligned}$$

Thus, for any integer x, if there exists some integer y such that $\nu_p(y) \equiv (d-1)\nu_p(x) \bmod d$ for all $p \in \{p_1, \ldots, p_k\}$, then $d \mid \nu_{p_i}(x) + \nu_{p_i}(y) = \nu_{p_i}(x+y)$ for all $1 \leq i \leq k$. We prove there are an infinite number of integers x with this property.

Observe that the set

$$S = \{(\nu_{p_1}(n), \ldots, \nu_{p_k}(n)) \bmod d \mid n \in \mathbb{N}\}$$

is finite, which means that we can choose a finite set of integers $Y = \{y_1, \ldots, y_l\}$ such that for any $(t_1, \ldots, t_k) \in S$,

$$(t_1, \ldots, t_k) \equiv (\nu_{p_1}(y), \ldots, \nu_{p_k}(y)) \bmod d$$

for some $y \in Y$. Letting $K = \prod_{i=1}^{k} p_i^{a_i}$ where $p_i^{a_i} > \max\{y_1, \ldots, y_l\}$, and z any positive integer, it follows that Kz is an integer where $u_{p_i}(Kz) > u_{p_i}(y)$ for all $y \in Y$. We have that

$$((\nu_{p_1}(Kz), \ldots, \nu_{p_k}(Kz)) \equiv (\nu_{p_1}(y_{j_1}), \ldots, \nu_{p_k}(y_{j_1})) \bmod d$$

for some $y_{j_1} \in Y$. Thus,

$$\begin{aligned}(\nu_{p_1}(Kz+y_{j_1}), \ldots, \nu_{p_k}(Kz+y_{j_1})) &\equiv (2\nu_{p_1}(Kz), \ldots, 2\nu_{p_k}(Kz)) \\ &\equiv (\nu_{p_1}(y_{j_2}), \ldots, \nu_{p_k}(y_{j_2})) \bmod d\end{aligned}$$

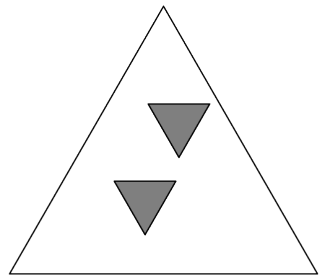
for some $y_{j_2} \in Y$. Continuing the process,

$$(\nu_{p_1}(Kz+y_{j_2}), \ldots, \nu_{p_k}(Kz+y_{j_2})) \equiv (3\nu_{p_1}(Kz), \ldots, 3\nu_{p_k}(Kz)) \bmod d.$$

By induction, there is some $y \in Y$ such that $\nu_{p_i}(Kz+y) \equiv d\nu_{p_i}(Kz) \equiv 0 \bmod d$ for $1 \leq i \leq k$. Since z was an arbitrary positive integer, there are an infinite number of integers Kz that satisfy the conditions of the problem.

4. An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral

triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. (An example with $n = 2$ is drawn below.)



Prove that

$$n \leq \frac{2}{3}L^2.$$

Solution: (Joanna Weng)

The greatest number of small triangles of the correct orientation to pack into $\triangle$ is

$$\frac{\lfloor L \rfloor (\lfloor L \rfloor - 1)}{2},$$

the triangle number. This bound gives

$$n \leq \frac{\lfloor L \rfloor (\lfloor L \rfloor - 1)}{2} < \frac{2}{3}L^2.$$

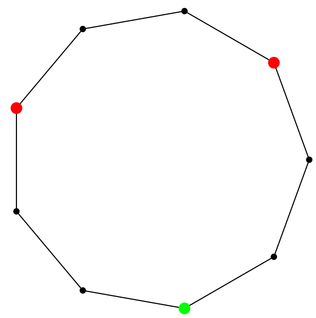
5. We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Solution: (Neo Huang)

(a) If n is odd, the regular n-gon works. Essentially, this is because if we choose any two distinct vertices A and B on the n-gon, one of the paths between A and B will contain and even number of vertices and the other an odd number. Choosing the "middle" vertex in the path with an odd number of vertices yields a point equidistant from A and B.



Now for the case where n is even. If $n = 4$, center a circle around one of the points O. Place the remaining three points on the circle such that they form an equilateral triangle. This forms a balanced set. We can always add two more points A, B to this construction and still get a balanced set by ensuring that AOB is an equilateral triangle. This works because if the two points we choose are on the circle, they are both equidistant from O. If one of the points is O and the other, A, is on the circle, then by construction, there is a point B on the circle such that AOB is an equilateral triangle.

(a) Part (a) shows that we can always form a balanced centre-free set if n is odd. It is impossible if n is even. Since there are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of points, and for each pair we must choose another point as that pair's "balancing point", a point P must be chosen as a balancing point at least $(n-1)/2$ times. Because n is even, the point P is actually chosen $n/2$ times. Since these $n/2$ pairs of points come from the remaining $n-1$ points other than P, at least one point A appears in two of the pairs, $\{A, B\}$ and $\{A, C\}$. But then we have that $PA = PB = PC$.