UBC Math Circle 2022 Problem Set 7

1. Given two triangles ΔABC and $\Delta A'B'C'$ with the same centroid, prove that one can construct a triangle with sides equal to the segments AA', BB', and CC'.

> **Solution:** (Young Lin)
>
> We first realize that if O is the centroid of some triangle ABC, then
>
> $$\vec{OA} + \vec{OB} + \vec{OC} = 0$$
>
> This is be easily shown (for example find a coordinate system, put one side of the triangle on the x-axis and do some algebraic manipulations.)
>
> We can also write the condition of three sides forming some triangle ABC by:
>
> $$\vec{AB} + \vec{BC} + \vec{CA} = 0$$
>
> The problem asks to form a triangle by AA', BB', CC' of arbitrary triangles ABC and $A'B'C'$, it suffices to show
>
> $$\vec{AA'} + \vec{BB'} + \vec{CC'} = 0$$
>
> $$-\vec{OA} + \vec{OA'} - \vec{OB} + \vec{OB'} - \vec{OC} + \vec{OC'} = 0 \tag{1}$$
>
> But since we know O is common centroid,
>
> $$\vec{OA} + \vec{OB} + \vec{OC} = \vec{OA'} + \vec{OB'} + \vec{OC'} = 0$$
>
> So we've showed (1), what we want to show. We are done.

2. Let M be the midpoint of the side BC in ΔABC. Let E and F be the tangent points of the incircle and the sides CA and AB, respectively. Let the angle bisectors of $\angle B$ and $\angle C$ intersect the line EF at X and Y, respectively. Prove that ΔMXY is equilateral if and only if $\angle A = 60°$.

> **Solution:** (Joanna Weng)
>
> From $\angle BDC = 180° - \frac{\angle B}{2} - \frac{C}{2}$, we have $\angle BDY = \frac{\angle B}{2} + \frac{C}{2} = 90° - \frac{\angle A}{2}$. From isoceles $\triangle AEF$, we have $\angle AFE = 90° - \frac{\angle A}{2}$. Thus, $\angle BDY = \angle FAD$. If Y lies inside segment EF, we directly conclude that $BDYF$ is cyclic by external angles; if Y lies outside segment EF, we have $\angle AFE = \angle BFY = \angle BDY$, so $BDFY$ is cyclic by corresponding angles. Then, $\angle BYD = \angle BFD = 90°$. (This result is called the intouch chord theorem!)

Similarly, we have $\angle CXD = 90°$.

In right $\triangle BYC$, M is the midpoint of the hypotenuse. Thus, $\angle MYC = \angle MCY = \angle YCA$, so $MY \parallel AC$. Similarly, $MX \parallel AB$.

We have $\triangle MYX \sim \triangle AFE$ by AA. With $AF = AE$ being tangents from a common point, $\triangle AFE$ is isoceles and equilateral iff $\angle A = 60°$; we conclude likewise for $\triangle MXY$.

3. Inside of convex quadrilateral $ABCD$ is a point M such that $\angle AMB = \angle ADM + \angle BCM$ and $\angle AMD = \angle ABM + \angle DCM$. Prove that

$$AM \cdot CM + BM \cdot DM \geq \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

Solution: (Arvin Sahami)

Perform the M inversion, the quadrilateral will be mapped to a parallelogram. In the new diagram, we should show that $MB.MD + MA.MC \geq AB.BC$; Assume that l, d are lines parallel to BC, AB respectively ,passing through M. l cuts AB, CD at X, Z and d cuts BC, AD at Y, T. Let $\alpha = \cos(\angle MTD)$ we will now compute all the lengths:
(Assume that $MX = x, MY = y, MZ = z, MT = t$.)

We need to show prove the following inequality:

$$\sqrt{x^2 + y^2 - 2xy\alpha} \cdot \sqrt{z^2 + t^2 - 2zt\alpha} + \sqrt{x^2 + t^2 + 2xt\alpha} \cdot \sqrt{z^2 + y^2 + 2yz\alpha} \\ \geq (x + z).(y + t)$$

There are non-negative real numbers $a, b, a + b = 1$ such that the above inequality could be represented as

$$\sqrt{a(x + y)^2 + b(x - y)^2} \cdot \sqrt{a(z + t)^2 + b(z - t)^2} \\ + \sqrt{b(x + t)^2 + a(x - t)^2} \cdot \sqrt{b(y + z)^2 + a(y - z)^2} \\ \geq (x + z).(y + t)$$

We can prove this by using Cauchy-Schwarz inequality.

4. The incircle of triangle $\triangle ABC$ with center I is tangent to the sides AB and BC at points C_1 and A_1, respectively. Let M be the midpoint of AC, and N be the midpoint of the arc ABC in the circumcircle of $\triangle ABC$. Let P be the projection of M over C_1A_1; show that I, P, N are collinear.

Solution: (Arvin Sahami)

Let T be the middle of the arc CA such that T and B lie on different sides of Ac. Let P be the intersection of IN and A_1C_1. Also let K be the intersection of IB and A_1C_1.

Observe that as all the points N, M, T lie on the perpendicular bisector of AC, we have $\angle AMN = \angle AMt = 90$

Furthermore, note that

$$\angle TAm = \angle TBC = \angle B/2, \angle TNA = \angle TBA = \angle B/2$$

Furthermore, it's easy to see that BI is the perpendicular bisector of A_1C_1 hence $BI \perp A_1C_1$, $\angle IBC = \angle B/2$ and $\angle IC_1B = 90$.

Now, since $\angle ANT = \angle C_1BI = \angle B/2$ and $\angle IC_1B = \angle TAN = 90$, we get that $\triangle IC_1B \sim \triangle TAN$.

Since $Am \perp TN$ and $IB \perp A_1C_1$, using the similarity obtained above, we get that $\frac{BK}{KI} = \frac{NM}{MT}$.

Now, since $\angle NBT = 180 - \angle NAT = 180 - 90 = 90$, and we also have $\angle C_1KB = 90$, we have $BN \parallel KC_1$. This implies $\frac{BK}{KI} = \frac{NP}{PI}$

Therefore we obtain $\frac{NP}{PI} = \frac{NM}{MT}$, hence $TI \parallel MP$. Now, since $\angle B/2 = \angle ABI = \angle ABT$, we have that B, I, T are collinear, and hence $BI \parallel MP$. Now since BI is the perpendicular bisector of A_1C_1 and hence $BI \perp A_1C_1$, we obtain $MP \perp A_1C_1$ as desired.

5. The incircle of triangle $\triangle ABC$ with center I has points of tangency D, E, and F. Let M be the foot of the perpendicular from D to EF, and let P be on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.

Solution: (Joanna Weng)

Let CI intersect BH at G so that $\angle BGC = 90°$ by definition of the orthocenter. Let BI intersect EF at Q. Let HI intersect EF at K. Let J be midpoint of EF.

By Menelaus theorem for $\triangle DMK$ and points P, J, H it suffices to show that

$$\frac{PM}{PN} \cdot \frac{HD}{HK} \cdot \frac{JK}{JM} = 1 \Longleftrightarrow \frac{HD}{HK} = \frac{JM}{JK}. \ (*)$$

From $AF = AE$ we get $IJ \perp EF$, so $JI \parallel DM$. Therefore

$$\frac{JM}{JK} = \frac{ID}{IK}.$$

We rewrite the relation $(*)$ as

$$\frac{HD}{HK} = \frac{ID}{IK}.$$

This is equivalent to showing that (D, I, K, H) is harmonical. Since $\triangle QGD$ is the orthic triangle of $\triangle HBC$, GI bisects $\angle DGK$. Since $\angle IGH = 90°$, GH is exterior angle bisector of $\angle DGK$ and the result follows.