UBC Math Circle 2022 Problem Set 8

1. Three numbers are chosen at random between 0 and 1. What is the probability that the difference between the greatest and the least is less than 1/3?

2. Find the probability such that when a polynomial in $\mathbb{Z}/2027\mathbb{Z}[x]$ having degree at most 2026 is chosen uniformly at random,

$$x^{2027} - x | P^k(x) - x \iff 2021 | k.$$

3. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

4. Prove that in a tournament with 799 teams, there exist 14 teams, that can be partitioned into groups in a way that all of the teams in the first group have won all of the teams in the second group.

5. A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.