UBC Math Circle 2022 Problem Set 8

1. Three numbers are chosen at random between 0 and 1. What is the probability that the difference between the greatest and the least is less than 1/3?

Solution: (Joanna Weng)

We approach this problem geometrically by observing that the desired probability is equal to the ratio of the volume in which every point satisfies the condition to the total possible volume.

Since the three numbers are not ordered, we can always pick out the smallest and for instance plot it on the z-axis. Then, the total possible region is (not the unit cube!) a square pyramid with unit square base in the xy-plane and volume 1/3. The points (x, y, z) that satisfy the condition also satisfy $x - z < \frac{1}{3}$ and $y - z < \frac{1}{3}$.

For $z > \frac{2}{3}$, all (x, y, z) in the possible region satisfy the condition. The volume of this part is therefore $\frac{1}{3} \cdot \left(\frac{1}{3}\right)^3 = \frac{1}{81}$. On the other hand, for $z < \frac{2}{3}$, we see that (x, y, z) in the possible region satisfy the condition if and only if $z \leq x, y \leq z + 1/3$. So at each z-plane section, the possible area equals $\left(\frac{1}{3}\right)^2$. The volume of this part is therefore $\frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{27}$. Hence, the total volume in which every point satisfies the condition is

$$\frac{1}{81} + \frac{2}{27} = \frac{7}{81}.$$

Thus, our ratio is

$$\frac{\frac{7}{81}}{\frac{1}{3}} = \frac{7}{27}.$$

2. Find the probability such that when a polynomial in $\mathbb{Z}/2027\mathbb{Z}[x]$ having degree at most 2026 is chosen uniformly at random,

$$x^{2027} - x | P^k(x) - x \iff 2021 | k.$$

Solution: (Oakley Edens)

Let $p = 2027$ and let $f : \mathbb{Z}/(p) \to \mathbb{Z}/(p)$ be an arbitrary function. Since p is prime, by Fermat's little theorem, the polynomial $f_a(x) = f(a)(1 - (x - a)^{2026})$ of degree 2026 is 0 for all $x \neq a$ and is $f(a)$ at $x = a$. It follows that $f(x) = \sum_{a \in \mathbb{Z}/(p)} f_a(x)$. Thus every function f is also a polynomial of degree at most 2026. Next, suppose f and g are polynomial functions of degree at most 2026 such that $f(a) = g(a)$ for all $a \in \mathbb{Z}/(p)$. Suppose $f - g \neq 0$. Since $\mathbb{Z}/(p)$ is a field, $f - g$

has at most $\deg(f-g)$ roots and it follows that $\deg(f-g) \geq 2027$ which is a contradiction. Thus $f = g$ and every polynomial of degree at most 2026 is a unique function from $\mathbb{Z}/(p)$ to itself. Thus choosing a random polynomial $P(x)$ of degree at most 2026 is equivalent to choosing a random function $f : \mathbb{Z}/(p) \to \mathbb{Z}/(p)$. The condition that $x^{2027} - x \mid P^k(x) - x$ is equivalent to $P^k(x) = x$ for all $x \in \mathbb{Z}/(p)$. This condition is preserved when we work with functions rather than polynomials. Hence let f be a function satisfying the desired property and define $\text{ord}_f(z)$ be the minimal k such that $f^k(z) = z$. The condition then implies that $\text{ord}_f(z)$ exists and $\text{lcm}_{z\in\mathbb{Z}/(p)}(\text{ord}_f(z)) = 43 \cdot 47$. Hence $\text{ord}_f(z) \in \{1, 43, 47, 2021\}$ or equivalently that every cycle $z \to f(z) \to \ldots \to f^k(z) = z$. Has $z \in \{1, 43, 47, 2021\}$. Next we work by cases. Note that if we have an m-cycle, there are $(m-1)!$ ways to orient it. Suppose there are a, b, c, d cycles of length $1, 43, 47, 2021$ respectively. We have that $a + 43b + 47c + 2021d = 2027$. If $d = 1$ then $b = c = 0$ and $a = 6$. There are then $\binom{2027}{6}(2020)!$ such functions. Next, suppose $d = 0$. The number of ways to orient n cycles of length m is $\frac{(nm)!}{n!(m!)^n}((m-1)!)^n = \frac{(nm)!}{n!m^n}$. Then the number of functions for any given values of a, b, c is $\frac{2027!}{a!(43b)!(47c)!}\frac{(43b)!}{b!43^b}\frac{(47c)!}{c!47^c} = \frac{2027!}{a!b!c!43^b47^c}$. Hence for $d = 0$ there are $\sum_{a+43b+47c=2027}\frac{2027!}{a!b!c!43^b47^c}$ functions satisfying the desired condition. Thus in total, there are $m = \frac{2027!}{6!2021!} + \sum_{a+43b+47c=2027}\frac{2027!}{a!b!c!43^b47^c}$ functions satisfying the conditions. Finally, there are 2027^{2027} total functions and thus the probability is $\frac{m}{2027^{2027}} \approx 9.65 \times 10^{-996}$.

3. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Solution: (Young Lin)

First of all for the two triangles to be congruent, since the points are all on the circle, if a set of three points can be rotated to cover another three points on the circle, the two triangles formed by each set of three points would be congruent. The proof will suffice if we can show for each of the four colours we can find a set of three points, such that each set of three points can be rotated to another. So this magic number N (number of points in each colour such then each triangle can be rotated to another) has to be bigger than or equal 3, and it reminds us to use pigeonhole principle.

We first consider all the points in Red, and all the non-identity rotations (that is rotate by $180° \cdot \frac{n}{432}$ where $n = 1, \cdots, 431$). There are 431 possibilities. There are 108 red points and green points, so among all the rotations the sum of instances a red point will land on a green point is 108^2. Denote a_n to be the number of red points that lie on green points, we have $\sum_{i=1}^{431} a_i = 108^2$. Then since $\lceil \frac{108^2}{431} \rceil = 28$, we know there is at least one set of 28 red points that will rotate to a set of 28 green points.

By the same reasoning $\lceil \frac{108\cdot 28}{431} \rceil = 8$ gives we can pick 8 from 28 green points to rotate to 8 blue points, and $\lceil \frac{108\cdot 8}{431} \rceil = 3$ shows there are three points from each of the other colours that can rotate to 3 yellow points. So we can just pick 3 points from each of the colours (unrotate the 3 yellow points if you wish), connect the lines and they would be congruent triangles. And the proof is complete.

4. Prove that in a tournament with 799 teams, there exist 14 teams, that can be partitioned into groups in a way that all of the teams in the first group have won all of the teams in the second group.

Solution: (Young Lin)

We will use Jensen's inequality which says for a real convex function ϕ, and numbers $x_1, \cdots, x_n$ in its domain,

$$\phi\left(\frac{\sum x_i}{n}\right) \leq \frac{\sum \phi(x_i)}{n}.$$

Let the ith team ($i = 1, \cdots 799$) win c_i games, then clearly we have $\sum_{i=1}^{799} c_i = \binom{799}{2}$. Then we randomly choose a set of 7 teams. Notice $\binom{c_i}{7}$ is the number of sets of 7 teams that are beaten by team i. Then applying Jensen's inequality (for $\phi(x) = \binom{x}{7}$ which is clearly convex)

$$\sum_{i=1}^{799} \binom{c_i}{7} \geq 799 \cdot \binom{\frac{\sum c_i}{799}}{7} = 799 \cdot \binom{399}{7}.$$

In the set of 7 teams (call it S) from this set of teams, we know $|S| = \binom{799}{7}$. Since

$$\frac{\sum_{i=1}^{799} \binom{c_i}{7}}{\binom{799}{7}} \geq \frac{799 \cdot \binom{399}{7}}{\binom{799}{7}} = \frac{396 \cdot 395 \cdot 394 \cdot 393}{2^3 \cdot 797 \cdot 795 \cdot 793} > 6$$

we know that the number of teams beaten by all teams can cover S more than 6 times. Take the ceiling we get that there would be at least 1 set of 7 teams that would be beaten by at least 7 other teams (think pigeonhole principle), and that gives the 14 teams we wanted. The proof is complete.

5. A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may

add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

Solution:

Assume without loss of generality that all pairs (x, x) on the blackboard occur only when $x > 0$. Let S be the set of all absolute values of all integers on the blackboard. For every $x \in S$, choose to erase all occurrences of x on the blackboard with probability p, where $p \geq 1-p$, or all occurrences of $-x$ on the blackboard with probability $1-p$. Then observe that for distinct $a, b \in S$ we get one point for each (on the blackboard)

$$\begin{aligned}
(a, a) &\text{ with probability } p, \\
(a, -a) \text{ or } (-a, a) &\text{ with probability } 1, \\
(a, b) &\text{ with probability } 1-(1-p)^2 \geq 1-p^2, \\
(a, -b) \text{ or } (-a, b) &\text{ with probability } 1-p(1-p) \geq 1-p^2, \\
(-a, -b) &\text{ with probability } 1-p^2.
\end{aligned}$$

So the expected number of points scored is at least $68 \cdot \min\{p, 1-p^2\}$, which is maximal when $p = 1-p^2$. Choosing this value for p, we find that $68p > 42$, and so N is at least 43.

Now we demonstrate a case where one can score 43 points and 43 is the maximum one can score. Consider a blackboard consisting of the following two kinds of pairs.

First kind: five of each pairs (i, i) where $1 \leq i \leq 8$. (40 total pairs.)

Second kind: one of each pair (i, j) where $-8 \leq i < j \leq -1$. (28 total pairs.)

Since $|S| = 8$, one can choose to erase $0 \leq n \leq 8$ of the first kind of pairs and $8-n$ of the second kind of pairs giving a total score of

$$5n + \binom{8}{2} - \binom{n}{2} = 5n + 28 - \binom{n}{2} \leq 43.$$

Equality can be achieved when $n = 5$ or $n = 6$.