UBC Math Circle 2022 Problem Set 9

1. Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n}(a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences a_1, a_2, a_3, $\ldots$ of nonnegative real numbers satisfying

$$\sum_{n=1}^{\infty} a_n = 1.$$

Solution: (Young Lin)

We know the AM-GM inequality:

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i.$$

always holds for all natural n if the x_i's are all positive. Then

$$n\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \sum_{i=1}^{n} x_i.$$

This seems to be a natural bound for the summands in S, giving

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n}\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \sum_{n=1}^{\infty} \frac{1}{2^n}\sum_{i=1}^{n} a_i.$$

Now we don't know how to proceed. In order for the equality to hold, we need all a_i's to be equal. But infinitely of them add up to 1, that's not possible.

Thinking of sequences, what are the common ones we know? Arithmetic? Geometric? Or just picking some random numbers? Is there any choice that would naturally sum up to 1 and also make AM-GM inequality bound optimally? GEOMETRIC SEQUENCES! Being an educated guess, we set

$$a_n = ar^{n-1}$$

And we have

$$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r} = 1$$

So $a = 1 - r$.

By the educated guess we start our bound by

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{n}{2^n} \left(\prod_{i=1}^{n} \frac{a_i}{r^i} r^i \right)^{\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{n}{2^n} \left(\prod_{i=1}^{n} \frac{a_i}{r^i} \right)^{\frac{1}{n}} r^{\frac{n(n+1)}{2n}}$$
$$= \sum_{n=1}^{\infty} \frac{n}{2^n} r^{\frac{n+1}{2}} \left(\prod_{i=1}^{n} \frac{a_i}{r^i} \right)^{\frac{1}{n}}.$$

Now we perform AM-GM inequality on the product term (this would be optimally bound since everything in the bracket $\frac{a_i}{r^i}$ would be equal (by it is a geometric sequence)). So

$$S \leq \sum_{n=1}^{\infty} \frac{1}{2^n} r^{\frac{n+1}{2}} \sum_{i=1}^{n} \frac{a_i}{r^i} = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{2^n} a_i r^{(n+1)/2-i}.$$

Now we can interchange the double sum $\sum_{n=1}^{\infty} \sum_{i=1}^{n}$ into $\sum_{i=1}^{\infty} \sum_{n=i}^{\infty}$ (we can see it by drawing a picture), and no limiting issues occur if the latter exists. Then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{2^n} a_i r^{(n+1)/2-i} = \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} \frac{1}{2^n} a_i r^{(n+1)/2-i} = \sum_{i=1}^{\infty} \frac{a_i}{r^i} r^{1/2} \sum_{n=i}^{\infty} (\frac{r^{1/2}}{2})^n$$

$$= \sum_{i=1}^{\infty} \frac{a_i}{r^i} r^{1/2} \frac{\frac{r^{i/2}}{2^i}}{1 - \frac{r^{1/2}}{2}} = \frac{r^{1/2}}{1 - \frac{r^{1/2}}{2}} \sum_{i=1}^{\infty} \frac{a_i}{r^i} \frac{r^{i/2}}{2^i} = \frac{r^{1/2}}{1 - \frac{r^{1/2}}{2}} \sum_{i=1}^{\infty} a_i (\frac{1}{2\sqrt{r}})^i.$$

And the sum would just be 1 if $\frac{1}{2\sqrt{r}}$ is 1, or $r = \frac{1}{4}$.

So essentially we've shown we can bound $S \leq \frac{r^{1/2}}{1 - \frac{r^{1/2}}{2}} = \frac{2}{3}$ if $r = \frac{1}{4}$ (as a general bound since we didn't put extra conditions on S when bounding it) and $a_n = ar^{n-1}$ a geometric sequence, but here we go among the derivation we only used 1 inequality, namely, the AM-GM inequality which is guaranteed to hold equal since we have an inequality (so this is the evidence that we can get to this bound). So at the end by picking $a_n = (1 - \frac{1}{4})(\frac{1}{4})^{n-1}$ we can achieve maximum S which is $\frac{2}{3}$.

2. Prove that the orthocenter, the centroid, and the circumcenter of any triangle are collinear. This line is the Euler Line of the triangle. Prove also that the distance from the centroid to the orthocenter is twice its distance from the circumcenter.

Solution: (Joanna Weng)

In $\triangle ABC$, let the midpoints of sides AB, BC, CA be F, D, E respectively.

Existence of the centroid: show that the three medians of a triangle concur.

Let $G = AD \cap CF$. We have that FD is a mid-segment of $\triangle ABC$: $FD \parallel AC$ and $2FD = AC$. Then, $\triangle ACG \sim \triangle DFG$ by AA. Thus,

$$\frac{GC}{FG} = \frac{GA}{DG} = \frac{AC}{FD} = 2.$$

Let $G' = CF \cap BE$. Similarly, $\frac{G'C}{FG'} = 2$. Both G and G' partition segment FC in the same ratio so $G = G'$. We have all medians concurring at a G, which we now call the centroid. We also have

$$\frac{GC}{FG} = \frac{GA}{DG} = \frac{GB}{EG} = 2.$$

Existence of the circumcenter: show the existence of a point O that is equidistant to points A, B, C (by constructing it).

If $AO = BO$, then $\angle OAB = \angle OBA$ and $\triangle OAD \cong \triangle OBD$ by SAS. Since $\angle ODA = \angle ODB$ and $\angle ODA + \angle ODB = 180°$, we have $\angle ODA = \angle ODB = 90°$ and OD is the perpendicular bisector of segment AB.

Let the intersection of perpendicular bisectors of AB, BC be O'. We have $AO' = BO'$ and $BO' = CO'$, so the required equidistant O exists in exactly one place and we now call it the circumcenter.

Existence of the orthocenter: show that the three altitudes of a triangle concur.

Note that O is the orthocenter of $\triangle DEF$. Similarly, construct points X, Y, Z such that $XB = YB = AC, XA = ZA = BC, YC = ZC = AB$. The orthocenter of $\triangle ABC$ is the point that is equidistant to the points X, Y, Z, which exists as shown above. Let the orthocenter be H.

Altogether

There exists a homothecy about G of factor $\frac{1}{2}$ that takes $\triangle ABC$ to $\triangle DEF$. O is taken to H.

3. Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ that satisfies the following conditions:

    1. For any natural number m, $f(m) \leq 3m$.

2. $v_2(m+n) = v_2(f(m)+f(n))$ for any two natural numbers m, n.[1]

Show that for any natural number $a \in \mathbb{N}$, there exists a unique number $b \in \mathbb{N}$ such that $f(b) = 3a$.

Solution: (Navid Safaei)

Letting $x = y$ we obtain $v_2(f(x)) = v_2(x)$. Then by putting $v_2(a) = k > 0$ we can consider $g(x) = \frac{f(2^k x)}{2^k}$ to reduce to all odd a. So we can assume that a is odd.

If $x - y$ is not divisible by 2^k we prove that $f(x) - f(y)$ is not divisible, too. Indeed, if $z \equiv -x \bmod 2^k$. It also follows that f is injective.

Let $2^{k-1} < 3a < 2^k$. Then $f(1), f(3), \ldots, f(2^k - 1)$ are pairwise distinct $\pmod{2^k}$. Hence, there is an odd $x < 2^k$ such that $f(x) \equiv 3a \pmod{2^k}$. If $f(x) \neq 3a$ then $f(x) > 2^k$ and $f(x) + f(2^k - x) \equiv 2^k \pmod{2^{k+1}}$. Thus, $f(x) + f(2^k - x) \geq 3 \cdot 2^k$. On the other hand,

$$f(x) + f(2^k - x) \leq 3(x + 2^k - x) = 3 \cdot 2^k.$$

Hence, $f(x) = 3x$. Thus $x \equiv a \pmod{2^k}$. That is, $x = a$ and so $f(x) = 3a$.

4. Let $V = \mathbb{Z}^n$ denote the n-dimensional integer lattice and let $\{w_1, w_2, \ldots, w_n\} \subset V$ be a set of n linearly independent integer vectors. Define $W \subset V$ to be the set of all integer linear combinations of the elements of $\{w_1, w_2, \ldots, w_n\}$. Construct a set V/W with elements of the form $v + W$ for $v \in V$ such that $u + W = v + W$ iff $u - v \in W$. Prove that $|V/W|$ is precisely the volume of the parallelotope P spanned by the vectors $\{w_1, w_2, \ldots, w_n\}$. (Hint: First prove that the volume of P is the sum $\sum_{i=0}^{n} \frac{|m_i|}{2^i}$ where m_i is the set of lattice points in an $n-i$-face of P but not in any $n-i-1$-face of P. For example, if $n = 3$ then m_0 is the set of lattice points inside the parallelopiped but not on any face of P, m_1 is the set of lattice points on a face of P but not on an edge of P, m_2 is the set of lattice points on an edge of P but not on any vertex of P, and m_3 is the set of lattice points on a vertex of P. Next show that $\sum_{i=0}^{n} \frac{|m_i|}{2^i} = |V/W|$.)

Solution: (Oakley Edens)

Let $\{w_1, w_2, \ldots, w_n\} \subset V$, W be the set of all integer linear combinations of the $\{w_i\}$ (i.e. the lattice spanned by the $\{w_i\}$) and P be the parallelotope spanned by the $\{w_i\}$. As in the problem, we define m_i to be the set of lattice points on an $n-i$ face of P but not on any $n-i-1$ faces of P. Let k be an arbitrary positive integer and P' a parallelotope spanned by the vectors $\{kw_1, kw_2, \ldots, kw_n\}$ so that k^n copies of P can be packed inside P'. Let m_i' be the set of lattice points on an

[1] $v_2(n)$ is the exponent of 2 in the factorization of n.

$n-i$ face of P' but not on any $n-i-1$ faces of P'. Finally, let $|P|$ and $|P'|$ denote the volumes of P and P' respectively. Note that $|m'_1|+|m'_2|+\ldots+|m'_n| = O(k^{n-1})$ and $|P'| = k^n|P| = |m'_0| + O(k^{n-1})$. Looking at the contributions to m'_0 we find $k^n m_0$ lattice points coming from the interiors of the k^n copies of P inside P', $\frac{k^n}{2}m_1 + O(k^{n-1})$ lattice points coming from the the lattice points of the $n-1$-faces of the k^n copies of P inside P' (the factor of $\frac{1}{2}$ comes from the fact that every $n-1$-face of a copy of P in the interior of P' is shared by exactly two distinct copies). Continuing as above, we get $\frac{k^n}{2^i}m_i + O(k^{n-1})$ lattice points from the points on $n-i$ faces but not on any $n-i-1$ faces of the k^n copies of P inside P'. Thus $k^n|P| = |P'| = |m'_0| + O(k^{n-1}) = k^n m_0 + \frac{k^n}{2}m_1 + \ldots + \frac{k^n}{2^n}m_n + O(k^{n-1})$. Dividing by k^n gives that $|P| = \sum_{i=0}^{n}\frac{|m_i|}{2^i} + O(k^{-1})$. Letting k go to infinity gives the desired equality.

Next, since the $\{w_i\}$ are linearly independent, they form a basis for $\mathbb{Q}^n$. In this basis, P is given by $[0,1]^n$ (i.e the Cartesian product of the interval). Let x be a lattice point inside P and $\overline{x} \in [0,1]^n$ the image of x in the basis $\{w_i\}$. Define $f(\overline{x})$ to be the number of coordinates of $\overline{x}$ which are either 0 or 1. It is not difficult to see that $x \in m_{f(\overline{x})}$. Finally, let u, v be lattice points, $\overline{u}$ and $\overline{v}$ their images in the basis $\{w_i\}$. Clearly $u+W = v+W$ iff $\overline{u}-\overline{v} \in \{0,1\}^n$. Suppose $u \in m_i$ and $v \in m_j$. If $i \neq j$ then from the above, we see that $\overline{u}, \overline{v}$ have a different number of coordinates which are either 0 or 1. It follows since $\overline{u}, \overline{v} \in [0,1]^n$ that $\overline{u}-\overline{v} \notin \{0,1\}^n$. If $i = j$ then $\overline{u}-\overline{v} \in \{0,1\}^n$ if only if all of the coordinates of $\overline{u}, \overline{v}$ which are neither 0 nor 1 are identical. Thus $n-i$ of the coordinates of $\overline{u}-\overline{v}$ are 0 and the remaining i can be chosen to be either 0 or 1. It follows that there are 2^i possibilities for $\overline{u}-\overline{v}$. It follows that the image of m_i in V/W has order $\frac{m_i}{2^i}$ (we over-count everything by a factor 2^i). Since every element of V/W corresponds to some lattice point inside P, we conclude that $|V/W| = \sum_{i=0}^{n}\frac{m_i}{2^i} = |P|$.

Here is an alternative (and much shorter proof) for those who know some abstract algebra. Let A be the matrix whose columns are the vectors $\{w_i\}$. The $\mathbb{Z}$-module W is given by $A\mathbb{Z}^n$ and $|V/W|$ is then the order of the $\mathbb{Z}$-module $\mathbb{Z}^n/A\mathbb{Z}^n$. Using the structure theorem for finitely generated modules over a principal ideal domain, $\mathbb{Z}^n/A\mathbb{Z}^n \cong \mathbb{Z}/(d_1) \oplus \ldots \oplus \mathbb{Z}/(d_n)$ and $|\mathbb{Z}^n/A\mathbb{Z}^n| = d_1 \ldots d_n$ since the d_i are nonzero (this follows from the fact that A is full rank). According to the Smith normal form, this will be the gcd of all $n \times n$-minors of A. This is simply $\det(A) = |P|$.

5. Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)

Solution:

Fix $\varepsilon > 0$, and let G be a graph on v vertices with at least $(1+\varepsilon)v$ edges. Suppose now that all simple cycles of G have pairwise distinct lengths. It suffices to show that v cannot be greater than some positive integer.

Since each simple cycle of G contains at most v vertices and no two distinct simple cycles of G can have equal lengths, it follows that G can have at most v distinct simple cycles. Now we find a lower bound for the distinct simple cycles of G. Let T be a spanning forest of G. Next, let A be the set of edges of T and let B be the set of remaining edges of G. Of course $|A| \leq v - c$, where c is the number of components of G. Then observe that $|B| \geq (1+\varepsilon)v - |A| \geq (1+\varepsilon)v - (v-c) = c + \varepsilon v > \varepsilon v$. Now since adding an edge to T produces a unique simple cycle, we see that for each $b \in B$, there is a unique simple cycle in G that contains b; let $C(b)$ denote this simple cycle. So there are $|B|$ distinct simple cycles $C(b)$, and since all simple cycles of G have pairwise distinct lengths, it follows that

$$\sum_{b \in B} (|C(b)| - 1) \geq 2 + 3 + \cdots + (|B| + 1) = \frac{|B|(|B|+3)}{2} \geq \frac{|B|^2}{2} > \varepsilon^2 v^2/2.$$

But $|A| \leq v$, and so some edge $e \in A$ occurs in $n > \varepsilon^2 v^2/(2v) = \varepsilon^2 v/2$ distinct cycles $C(b)$. Consider distinct $b_1, b_2 \in B$ such that e is an edge of $C(b_1)$ and $C(b_2)$. Observe that $C(b_1) \cap C(b_2)$ is a path in T that contains e, and that b_1 and b_2 are not edges in $C(b_1) \cap C(b_2)$. Then by deleting the internal vertices of this path in $C(b_1) \cup C(b_2)$, we obtain a new simple cycle $C(b_1, b_2)$ that contains both b_1 and b_2 that is not among the simple cycles $C(b)$. Furthermore, this new simple cycle is uniquely determined by the choice of b_1 and b_2, i.e., $\{b_1, b_2\} \mapsto C(b_1, b_2)$ is injective. Hence, as a lower bound, G must contain at least

$$n + \binom{n}{2} = \frac{n(n+1)}{2} \geq \frac{n^2}{2} > \frac{\varepsilon^4 v^2}{8}$$

distinct simple cycles. But of course there is some positive integer u such that

$$u \leq \frac{\varepsilon^4 u^2}{8}.$$

Hence, we have shown that $v < u$, which completes the proof.