UBC Math Circle 2023 Problem Set 1

Problem 1.

The sequence given by $x_0 = a, x_1 = b$, and

$$x_{n+1} = \frac{1}{2}(x_{n-1} + \frac{1}{x_n})$$

is periodic.

Prove that $ab = 1$.

Problem 2. Chebyshev polynomials $T_n(x), U_n(x)$ are defined by $T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ and $U_0(x) = 1, U_1(x) = 2x, U_{n+1}(x) = xU_n(x) - U_{n-1}(x)$, and they are determined by the equalities

$$\cos(n\theta) = T_n(\cos(\theta)), \quad \frac{\sin((n+1)\theta)}{\sin\theta} = U_n(\cos\theta)$$

For $n \geq 1$, try to prove

$$\frac{T_n(x)}{\sqrt{1-x^2}} = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}$$

$$U_n(x)\sqrt{1-x^2} = \frac{(-1)^n(n+1)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}$$

Problem 3.

Let $\mathbb{Q}[\zeta_5] = \{a_0 + a_1\zeta_5 + a_2\zeta_5^2 + a_3\zeta_5^3 + a_4\zeta_5^4 : a_i \in \mathbb{Q}\}$ and $\mathbb{Z}[\zeta_5] = \{a_0 + a_1\zeta_5 + a_2\zeta_5^2 + a_3\zeta_5^3 + a_4\zeta_5 : a_i \in \mathbb{Z}\}$, where ζ_5 is a primitive 5th root of unity. Note that $\mathbb{Q}[\zeta_5]$ is a field (equipped with $+$ and $\cdot$ from $\mathbb{C}$ it is closed under addition/multiplication and has additive/multiplicative inverses) while $\mathbb{Z}[\zeta_5]$ is a ring (it is closed under addition/multiplication, has additive inverses but not necessarily multiplicative inverses).

(a) Define $\sigma_1 : \mathbb{Q}[\zeta_5] \to \mathbb{Q}[\zeta_5]$ by $\sigma_1(z) = z$ and $\sigma_2 : \mathbb{Q}[\zeta_5] \to \mathbb{Q}[\zeta_5]$ by $\sigma_2(\sum_{i=0}^4 a_i\zeta_5^i) = \sum_{i=0}^4 a_i\zeta_5^{2i}$. Note the following properties of σ_i: $\sigma_i(z+w) = \sigma_i(z) + \sigma_i(w)$ and $\sigma_i(zw) = \sigma_i(z)\sigma_i(w)$. (These are two of the four field automorphisms on $\mathbb{Q}[\zeta_5]$, with the other two being $\overline{\sigma}_i$). We have a map $N : \mathbb{Q}[\zeta_5] \to \mathbb{R}$ given by $N(z) = |\sigma_1(z)\sigma_2(z)|^2$. (In fact you can check that $N(z) \in \mathbb{Q}$). Prove that for any $a, b \in \mathbb{Z}[\zeta_5]$ with $b \neq 0$, there exist $q, r \in \mathbb{Z}[\zeta_5]$ such that $a = qb + r$ and $N(r) < N(b)$.

(b) We call an element $p \in \mathbb{Z}[\zeta_5]$ prime if whenever $p \mid ab$ for $a, b \in \mathbb{Z}[\zeta_5]$, we have either $p \mid a$ or $p \mid b$. Use (a) to prove that every $z \in \mathbb{Z}[\zeta_5]$ may be written uniquely as $z = p_1^{a_1} \dots p_n^{a_n}$ where the p_i are prime and the $a_i \geq 1$ up to rearrangement and multiplication by unit elements u satisfying $N(u) = 1$ (that is two representations are considered equivalent if one can get from one to the other by rearranging terms and multiplying by units).

(c) Use (b) to prove that the equation $x^5 + y^5 = z^5$ has no solutions in nonzero integers.