UBC Math Circle 2023 Problem Set 2

Problem 1. Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n, b_n such that the polynomial $q_n(x) = x^n - a_n x - b_n$ is divisible by $p(x)$.

solution 1. Assume that we have found such numbers for every n. Then $q_{n+1}(x) - xq_n(x)$ must be divisible by $p(x)$. But

$$q_{n+1}(x) - xq_n(x) = x^{n+1} - a_{n+1}x - b_{n+1} - x^{n+1} + a_n x^2 + b_n x$$

$$= -a_{n+1}x - b_{n+1} + a_n(x^2 - 3x + 2) + 3a_n x - 2a_n + b_n x$$

$$= a_n(x^2 - 3x + 2) + (3a_n + b_n - a_{n+1})x - (2a_n + b_{n+1})$$

and this is divisible by $p(x)$ if and only if $3a_n + b_n - a_{n+1}, 2a_n + b_{n+1}$ are both equal to zero. This means that the sequences are uniquely determined by the recurrences $a_1 = 3, b_1 = -2, a_{n+1} = 3a_n + b_n, b_{n+1} = -2a_n$. The sequences exist and are uniquely defined by the initial condition.

Problem 2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001, \forall n \in \mathbb{N}$$

solution 2. We first try a function of the form $f(n) = n + a$. The relation from the statement yields $a = 667$, and hence $f(n) = n + 667$ is a solution. Let us show that this is the only solution.

Fix some positive integer n and define $a_0 = n$, and $a_k = f(f(\cdots(f(n)\cdots)))$, where the composition is taken k times, $k \geq 1$. The sequence $(a_k)_{k\geq 0}$ satisfies the inhomogeneous linear recurrence relation

$$a_{k+3} - 3a_{k+2} + 6a_{k+1} - 4a_k = 2001$$

A particular solution is $a_k = 667k$. The characteristic equation of the homogeneous recurrence $a_{k+3} - 3a_{k+2} + 6a_{k+1} - 4a_k = 0$ is

$$\lambda^3 - 3\lambda^2 + 6\lambda - 4 = 0$$

An easy check shows that $\lambda_1 = 1$ is a solution to this equation. Since $\lambda^3 - 3\lambda^2 + 6\lambda - 4 = (\lambda - 1)(\lambda^2 - 2\lambda + 4)$, the other two solutions are $\lambda_{2,3} = 1 \pm i\sqrt{3}$, that is, $\lambda_{2,3} = 2(\cos(\frac{\pi}{3}) \pm i\sin(\frac{\pi}{3}))$. It follows that the formula for the general term of a sequence satisfying the recurrence relation is

$$a_k = c_1 + c_2 2^k \cos(\frac{k\pi}{3}) + c_3 2^k \sin(\frac{k\pi}{3}) + 667k, \quad k \geq 0$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

If $c_2 > 0$, then $a_{3(2m+1)}$ will be negative for large m, and if $c_2 < 0$, then a_{6m} will be negative for large m. Since $f(n)$ can take only positive values, this implies that $c_2 = 0$. A similar argument shows that $c_3 = 0$. It follows that $a_k = c_1 + 667k$. So the first term of the sequence determines all others. Since $a_0 = n$, we have $c_1 = n$, and hence $a_k = n + 667k$, for all k. In particular, $a_1 = f(n) = n + 667$, and hence this is the only possible solution.

Problem 3. A polygon is called *convex* if all its internal angles are smaller than 180 degrees. Given a convex polygon, prove that one can find three distinct vertices A, P, Q, where PQ is a side of the polygon, such that the perpendicular from A to the line PQ meets the segment PQ.

solution 3. Let n be the number of vertices. Let PQ be the longest side of the plygon, label the vertices by $A_1, A_2, \dots, A_n$ such that $A_1 = P,\ A_n = Q$.

Let ℓ_1, ℓ_2 be the lines perpendicular to A_1A_n passing through A_1, A_n respectively.

Let S be the region between ℓ_1, ℓ_2. Note that for $2 \leq i \leq n-1$ if A_i lies on S, then the perpendicular from A_i to A_1A_n would lie on the segment A_1A_n and we are done.

Hence we may assume none of the A_i's lies in S for $2 \leq i \leq n-1$.

Color all the points lying on the same side of ℓ_2 as A_1 red , and the points lying on the same side of ℓ_1 as A_n blue.

Note that since no point lies in S (other than A_1, A_n), each point receives exactly one color.

Now moving along the path A_iA_{i+1}, starting from A_1 we eventually reach A_n. But A_1 is red and A_n is blue, hence we can consider the first index such that we have crossed from a red point to a blue point. Let this index be i, hence A_i is red and A_{i+1} is blue. Note that since $3 \leq n$ we know $(A_i, A_{i+1}) \neq (A_1, A_n)$ hence A_i, A_{i+1} are on different sides of S with at least one of them not lying in it.

It now follows that $A_iA_{i+1} > A_1A_n$, contradicting that A_1A_n was the largest side.

Hence there is point whose altitude to A_1A_n falls on the segment A_1A_n, as desired. $\square$