UBC Math Circle 2023 Problem Set 3

Problem 1. Show that $\frac{400!\times 399!\cdots 2!\times 1!}{200!}$ is a perfect square.

Problem 2. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

Problem 3. In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC in the tetrahedron is the intersection of the altitudes of $\triangle ABC$. Prove that

$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2)$.

For what tetrahedra does equality hold?

(hint: what can we say about angles ADB and ADC?)