UBC Math Circle 2023 Problem Set 3 Solutions

Problem 1. Show that $\frac{400!\times 399!\cdots 2!\times 1!}{200!}$ is a perfect square.

Proof. Rearranging the factors in the numerator

$$\frac{400!\cdot 399!\cdot\dots\cdot 2!\cdot 1!}{200!}=\frac{400^1\cdot 399^2\cdot\dots\cdot 2^{399}\cdot 1^{400}}{200!}$$

factoring out the factors with even exponent, we get

$$\frac{400!\cdot 399!\cdot\dots\cdot 2!\cdot 1!}{200!}=\frac{400^1\cdot 398^3\cdots 4^{397}\cdot 2^{399}}{200!}\cdot(399^2\cdot 397^4\cdots 3^{398}\cdot 1^{400})$$
$$=\frac{400^1\cdot 398^3\cdots 4^{397}\cdot 2^{399}}{200!}\cdot(399^1\cdot 397^2\cdots 3^{199}\cdot 1^{200})^2$$

Notice how the factors with odd exponent are all even, we may do the following operations

$$\frac{400!\cdot 399!\cdot\dots\cdot 2!\cdot 1!}{200!}=\frac{400^1\cdot 398^3\cdots 4^{397}\cdot 2^{399}}{200!}\cdot(399^1\cdot 397^2\cdots 3^{199}\cdot 1^{200})^2$$
$$=\frac{(2\cdot 200)^1\cdot(2\cdot 199)^3\cdots(2\cdot 2)^{397}\cdot(2\cdot 1)^{399}}{200!}\cdot(399^1\cdot 397^2\cdots 3^{199}\cdot 1^{200})^2$$
$$=(2^1\cdot 200^0)\cdot(2^3\cdot 199^2)\cdots(2^{397}\cdot 2^{396})\cdot(2^{399}\cdot 1^{398})\cdot(399^1\cdot 397^2\cdots 3^{199}\cdot 1^{200})^2$$
$$=\prod_{i=0}^{199}2^{2i+1}\cdot(200^0\cdot 199^1\cdots 2^{198}\cdot 1^{199})^2(399^1\cdot 397^2\cdots 3^{199}\cdot 1^{200})^2$$

Since the exponent of 2 is the sum of 200 odd numbers and the sum of odd numbers is even, we have the product of three perfect squares. Thus $\frac{400!\cdot 399!\cdots 2!\cdot 1!}{200!}$ is a perfect square as required. □

Problem 2. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

Proof. Let O be the circumcenter of ABC. From the assumption that $\angle ACB = 120°$ it follows that M is the midpoint of CO.

Let ω denote the circle with center in C and radius CO. This circle in the image of the circumcircle of ABC through the translation that sends O to C. We claim that N lies on ω.

Let us consider the triangles QNP and QMC. The angles in Q are equal. Since NP is parallel to MC (both lines are perpendicular to AB), it turns out that $\angle QNP = \angle QMC$, and hence the two triangles are similar. Since $QP = 2QC$, it follows that

$$NP = 2MC = CO$$

which proves that N lies on ω. □

Problem 3. In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC in the tetrahedron is the intersection of the altitudes of $\triangle ABC$. Prove that

$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2)$.

For what tetrahedra does equality hold?

(hint: what can we say about angles ADB and ADC?)

Proof. Let us show first that angles ADB and ADC are also right. Let H be the intersection of the altitudes of ABC and let CH meet AB at X. Planes CED and ABC are perpendicular and AB is perpendicular to the line of intersection CE. Hence AB is perpendicular to the plane CDE and hence to ED. So $BD^2 = DE^2 + BE^2$. Also $CB^2 = CE^2 + BE^2$. Therefore $CB^2 - BD^2 = CE^2 - DE^2$. But $CB^2 - BD^2 = CD^2$, so $CE^2 = CD^2 + DE^2$, so angle $CDE = 90°$. But angle $CDB = 90°$, so CD is perpendicular to the plane DAB, and hence angle CDA = $90°$. Similarly, angle $ADB = 90°$. Hence $AB^2 + BC^2 + CA^2 = 2(DA^2 + DB^2 + DC^2)$. But now we are done, because Cauchy's inequality gives $(AB+BC+CA)^2 = 3(AB^2+BC^2+CA^2)$. We have equality if and only if we have equality in Cauchy's inequality, which means $AB = BC = CA$. $\square$