UBC Math Circle 2023 Problem Set 4

Problem 1. Consider a circle of diameter AB and center O, and the tangent t at B. A variable tangent to the circle with contact point M intersects t at P. Find the locus of the point Q where the line OM intersects the parallel through P to the line AB.

Problem 2. Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \cdots$ such that $a_n \in \{1, 2, \cdots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1}$ (mod p) for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

Problem 3. Given is a finite set of spherical planets all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.