UBC Math Circle 2023 Problem Set 4 Solutions

Problem 1. Consider a circle of diameter AB and center O, and the tangent t at B. A variable tangent to the circle with contact point M intersects t at P. Find the locus of the point Q where the line OM intersects the parallel through P to the line AB.

Proof. We assume that the radius of the circle is equal to 1. Set the origin at B with BA the positive x-semiaxis and t the y-axis. If $\angle BOM = \theta$, then $BP = PM = \tan(\frac{\theta}{2})$. In triangle PQM, $PQ = \frac{\tan(\frac{\theta}{2})}{\sin\theta}$. So the coordinates of Q are

$$(\frac{\tan(\frac{\theta}{2})}{\sin\theta}, \tan(\frac{\theta}{2})) = (\frac{1}{1+\cos\theta}, \frac{\sin\theta}{1+\cos\theta})$$

The x and y coordinates are related as follows:

$$(\frac{\sin\theta}{1+\cos\theta})^2 = \frac{1-\cos^2(\theta)}{(1+\cos(\theta))^2} = \frac{1-\cos\theta}{1+\cos\theta} = 2\frac{1}{1+\cos\theta} - 1$$

Hence the locus of Q is the parabola $y^2 = 2x - 1$. □

Problem 2. Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \cdots$ such that $a_n \in \{1, 2, \cdots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1}$ (mod p) for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

Proof. We say that an ordered pair of integers (a, b) has property P if $a, b \in \{1, 2, ..., p-2\}$ and $a + b \neq p - 1$. We claim that if (a_1, a_2) has property P, then it is part of a unique infinite sequence $a_1, a_2, a_3, \cdots$ that satisfies the conditions of the problem. The justification uses the following lemma.

LEMMA 1. The conditions in the problem statement on a_n, a_{n+1}, a_{n+2} imply that

$$a_{n+2} \equiv -1 \pmod p \iff a_n + a_{n+1} \equiv -1 \pmod p$$
$$a_n \equiv -1 \pmod p \iff a_{n+1} + a_{n+2} \equiv -1 \pmod p$$

The first equivalence follows from $a_n(a_{n+2}+1) \equiv 1 + a_{n+1} + a_n \pmod p$, and the fact that we can divide by a_n modulo p since a_n is not a multiple of p. The second equivalence is proved similarly. □

If (a_n, a_{n+1}) has property P, then $a_n + a_{n+1} \not\equiv -1 \pmod p$. Since a_n and $1 + a_{n+1}$ are not multiples of p, there is a unique $a_{n+2} \in \{1, \cdots, p-1\}$ such that $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod p$, and $a_{n+2} \neq p - 1$ by the first part of the lemma. Further, by the second part of the lemma, $a_{n+1} + a_{n+2} \neq p - 1$. Thus, (a_{n+1}, a_{n+2}) has property P, and is uniquely determined by (a_n, a_{n+1}) and the conditions in the problem statement. Our claim follows by induction on n.

Next, suppose that the sequence $a_1, a_2, a_3, \cdots$ that satisfies the conditions of the problem. For $n \geq 1$, since a_n and a_{n+2} are not multiples of the prime number p, neither $a_n a_{n+2}$, so $a_{n+1} \not\equiv -1 \pmod p$. In particular, a_2, a_3, a_4 are not congruent to -1 modulo p. Then by the first part of the lemma, $a_1 + a_2$ and $a_2 + a_3$ are not congruent to -1 modulo p. Then by the second part of the lemma, $a_1 \not\equiv -1 \pmod p$. In particular, (a_1, a_2) must have property P.

Therefore, $f(p)$ is the number of ordered pairs (a_1, a_2) that have property P. There are $p - 2$ possible values of a_1, and for each such value, there are $p - 3$ values of a_2 consistent with property

P. Then $f(p) = (p-2)(p-3) = p^2 - 5p + 6$ and $f(p) \equiv p^2 + 1 \pmod 5$. Since $p \neq 5$ and p is a prime, p is congruent to $1, 2, 3, 4$ modulo 5. In all cases, $p^2 \equiv \pm 1 \pmod 5$, and the conclusion of the problem follows. $\square$

Problem 3. Given is a finite set of spherical planets all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

Proof. Choose a preferential direction in space, which defines the north pole of each planet. Next, define an order on the set of planets by saying that planet A is greater than planet B, $A < B$ if on removing all other planets from space, the north pole of B is invisible from A. The only case in which something can go wrong is that in which the preferential direction is perpendicular to the segment joining the centers of two planets. If this is not the case, then $<$ defines a total order on the planets. This order has a unique maximal element M. The north pole of M is only north pole not visible from another planet.

Now consider a sphere of the same radius as the planets. Remove from it all north poles defined by directions that are perpendicular to the axes of two of the planets. This is a set of area zero. For every other point on this sphere, there exists a direction in space that makes it the north pole, and for that direction, there exists a unique north pole on one of the planets that is not visible from the others. As such, the surface of the newly introduced sphere is covered by patches translated from the other planets. Hence the total area of invisible points is equal to the area of this sphere, which in turn is the area of one of the planets. $\square$