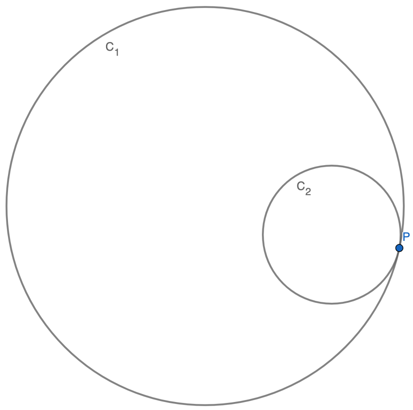
UBC Math Circle 2023 Problem Set 5

Problem 1. Two circles, $C1$ and $C2$, are internally tangent at point P; their interiors overlap. Construct $\triangle ABP$ with A on $C1$ and B on $C2$ such that its area is maximized. If the radii of $C1$ and $C2$ are respectively 3 and 1, what is this maximum area?



Problem 2. Show that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

Problem 3. Consider a function $f\colon A \to \mathbb{R}$ for which there is fixed $d \in \mathbb{N}$ satisfying the following:

For any $0 < \epsilon$ there is a polynomial $P\epsilon(x) \in \mathbb{R}[x]$ such that $\deg(P\epsilon) \leq d$, and for all $x \in A, |P_\epsilon(x) - f(x)| < \epsilon$.

Show that there is a fixed polynomials $P(x) \in \mathbb{R}[x]$ such that $f(x) = P(x)$ for all $x \in A$, where

- $A = \mathbb{R}$.
- A is unbounded.
- (Harder) $A \subset \mathbb{R}$ is arbitrary.