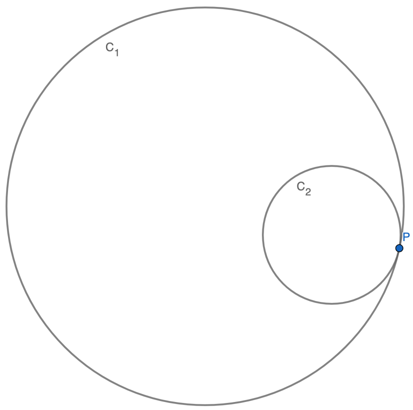
UBC Math Circle 2023 Problem Set 5

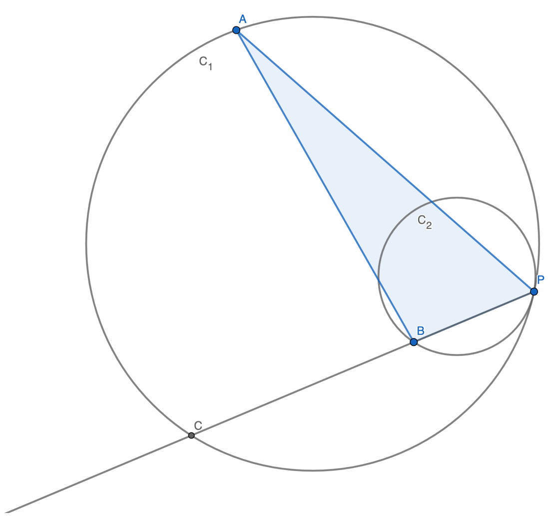
Problem 1. Two circles, $C1$ and $C2$, are internally tangent at point P; their interiors overlap. Construct $\triangle ABP$ with A on $C1$ and B on $C2$ such that its area is maximized. If the radii of $C1$ and $C2$ are respectively 3 and 1, what is this maximum area?



Proof. Note that there exists a homothety h centered at P of factor 3 that takes $C2$ to $C1$.

Extend PB to C on $C1$. Since C is the image of B under h, we have $PC = 3PB$. Thus, $[ACP] = 3[ABP]$ and now we just need to maximize $[ACP]$.

To maximize the area of a circumscribed triangle, we make it equilateral and have area $\frac{3\sqrt{3}R^2}{4}$ where R is the circumscribing radius. Since $\triangle ACP$ is a circumscribed triangle of $C1$, its maximum area is $\frac{27\sqrt{3}}{4}$. Thus, the maximum area of $\triangle ABP$ is $\frac{9\sqrt{3}}{4}$.



□

Problem 2. Show that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

Proof. Notice by binomial theorem we have

$$S_n = \frac{1}{4}[(2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}]$$

The fact that $S_n = (k-1)^2 + k^2$ for some positive integer k is equivalent to

$$2k^2 - 2k + 1 - S_n = 0$$

View this as q quadratic equation in k, its determinant is

$$\Delta = 4(2S_n - 1) = 2[(2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1} - 2]$$

Is this a perfect square? The numbers $(2+\sqrt{3})$ and $(2-\sqrt{3})$ are one the reciprocal of the other, and if they were squares, we would have a perfect square. In fact, $(4 \pm 2\sqrt{3})$ are squares of $(1 \pm \sqrt{3})$. We find that

$$\Delta = (\frac{(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}}{2^n})^2$$

Solving the quadratic equation, we find that

$$k = \frac{1}{2} + \frac{(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}}{2^{2n+2}}$$

$$= \frac{1}{2} + \frac{1}{4}[(1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n]$$

This is clearly a rational number, but is it an integer? The numbers $2+\sqrt{3}$ and $2-\sqrt{3}$ are the roots of the equation

$$\lambda^2 - 4\lambda + 1 = 0$$

which can be interpreted as the characteristic equation of a recursive sequence $x_{n+1} - 4x_n + x_{n-1} = 0$, for which we can solve for the general formula as

$$x_n = (1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n$$

and we can also see that $x_0 = 2, x_1 = 10$. An induction based on the recurrence relation shows that x_n is divisible by 2 but not by 4. It follows that k is an integer and the problem is solved. □

Problem 3. Consider a function $f\colon A \to \mathbb{R}$ for which there is fixed $d \in \mathbb{N}$ satisfying the following:

For any $0 < \epsilon$ there is a polynomial $P\epsilon(x) \in \mathbb{R}[x]$ such that $\deg(P\epsilon) \leq d$, and for all $x \in A, |P_\epsilon(x) - f(x)| < \epsilon$.

Show that there is a fixed polynomials $P(x) \in \mathbb{R}[x]$ such that $f(x) = P(x)$ for all $x \in A$, where

- $A = \mathbb{R}$.
- A is unbounded.
- (Harder) $A \subset \mathbb{R}$ is arbitrary.

Proof. Let us state the hypothesis in a cleaner way, you may verify that it is equivalent to the following:

There exists a sequence of real polynomials $\{Q_n\}_{n\in\mathbb{N}}$ of degree at most d, such that for any $0 < \varepsilon$ there is $N \in \mathbb{N}$ such that for all $N \leq n, x \in A$ we have

$$|f(x) - Q_n(x)| < \varepsilon$$

(this is called uniform convergence)

Let us write

$$Q_n(x) = \sum_{i=0}^{d} q_{i,n}x^i$$

Let $A \subseteq \mathbb{R}$ be arbitrary, note that if $|A| \leq d+1$ we are done using Lagrange interpolation.
Hence we may assume $d+1 < |A|$. Pick $d+1$ distinct points $x_1, x_2 \ldots x_{d+1} \in A$.
Let M be the $(d+1) \times (d+1)$ Vandermonde matrix formed with these coefficients.
Note that using the Vandermonde determinant , since the coefficients are distinct, we conclude that M is invertible.

Further note that letting $f(x_k) = y_k$ (for $1 \leq k \leq d+1$) we have

$$\lim_{n\to\infty} M(q_{0,n}, q_{1,n}, \ldots, q_{d,n})^T = (y_1, \ldots, y_{d+1})^T$$

(where the limit is coefficient wise).

But since M^{-1} is a fixed matrix, we may multiply it by both sides to get

$$\lim_{n\to\infty} (q_{0,n}, q_{1,n}, \ldots, q_{d,n})^T = M^{-1}(y_1, \ldots, y_{d+1})^T$$

Hence letting

$$(p_0, p_1, \ldots, p_d)^T = M^{-1}(y_1, \ldots, y_{d+1})$$

for all $0 \leq i \leq d$ we have

$$\lim_{n\to\infty} q_{i,n} = p_i \qquad (\star)$$

Now take the polynomial

$$P(x) = \sum_{i=0}^{d} p_i x^i$$

Consider any fixed $x \in A$, note that using $(\star)$ it follows that

$$\lim_{n\to\infty} Q_n(x) = P(x)$$

but the hypothesis also yields (why?)

$$\lim_{n\to\infty} Q_n(x) = f(x)$$

hence

$$P(x) = f(x)$$

for all $x \in A$, a desired. □