UBC Math Circle 2023 Problem Set 6

Problem 1. Let $x_1, x_2, \cdots, x_n$, $n \geq 2$, be positive numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$(1 + \frac{1}{x_1})(1 + \frac{1}{x_2}) \cdots (1 + \frac{1}{x_n}) \geq (n+1)^n$$

Problem 2. Consider the sequences $(a_n)_n, (b_n)_n$ defined by $a_1 = 3, b_1 = 100, a_{n+1} = 3^{a_n}, b_{n+1} = 100^{b_n}$. Find the smallest number m for which $b_m > a_{100}$.

Problem 3. Define the sequence $(a_n)_n$ recursively by $a_1 = 2, a_2 = 5$ and

$$a_{n+1} = (2 - n^2)a_n + (2 + n^2)a_{n-1} \quad n \geq 2$$

Do there exist indices p, q, r such that $a_p \cdot a_q = a_r$?