UBC Math Circle 2023 Problem Set 6

Problem 1. Let $x_1, x_2, \cdots, x_n$, $n \geq 2$, be positive numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$(1+\frac{1}{x_1})(1+\frac{1}{x_2})\cdots(1+\frac{1}{x_n}) \geq (n+1)^n$$

Proof. If $x_i < x_j$ for some i, j, increase x_i and decrease x_j by some number a, $0 < a \leq x_j - x_i$. We need to show that

$$(1+\frac{1}{x_i+a})(1+\frac{1}{x_j-a}) < (1+\frac{1}{x_i})(1+\frac{1}{x_j})$$

or

$$\frac{(x_i+a+1)(x_j-a+1)}{(x_i+a)(x_j-a)} < \frac{(x_i+1)(x_j+1)}{x_i x_j}$$

where all the denominators are positive, so after multiplying out and cancelling terms, we obtain the equivalent inequality

$$-ax_i^2 + ax_j^2 - a^2x_i - a^2x_j - ax_i + ax_j - a^2 > 0$$

This can be rewritten as

$$a(x_j - x_i)(x_j + x_i + 1) > a^2(x_j + x_i + 1)$$

which is true, since $a < x_j - x_i$. Starting with the smallest and the largest of the numbers, we apply the trick and make one of the numbers equal to $\frac{1}{n}$ by decreasing the value of the expression. Repeating, we can decrease the expression to one in which all numbers are equal to $\frac{1}{n}$. The value of the latter expression is $(n+1)^n$. This concludes the proof. □

Problem 2. Consider the sequences $(a_n)_n, (b_n)_n$ defined by $a_1 = 3, b_1 = 100, a_{n+1} = 3^{a_n}, b_{n+1} = 100^{b_n}$. Find the smallest number m for which $b_m > a_{100}$.

Proof. We need to determine m such that $b_m > a_n > b_{m-1}$. It seems that the difficult part is to prove an inequality for the form $a_n > b_m$,which reduces to $3^{a_{n-1}} > 100^{b_{m-1}}$, or $a_{n-1} > (\log_3 100)b_{m-1}$. Iterating, we obtain $3^{a_{n-2}} > (\log_3 100)100^{b_{m-2}}$, that is

$$a_{n-2} > \log_3(\log_3 100) + (\log_3 100)b_{m-2}$$

Seeing this we might suspect that an inequality of the form $a_n > u + vb_m$, holding for all n with some fixed u and v, might be useful in the solution. From such an inequality we would derive $a_{n+1} = 3^{a_n} > 3^u(3^v)^{b_m}$. If $3^v > 100$, then $a_{n+1} > 3^u b_{m+1}$, and if $3^u > u + v$, then we would obtain $a_{n+1} > u + vb_{m+1}$, the same inequality as the one we started with, but with $m+1$ and $n+1$ instead of m and n.

The inequality $3^v > 100$ holds for $v = 5$, and $3^u > u + 5$ holds for $u = 2$. Thus $a_n > 2 + 5b_m$ implies $a_{n+1} > 2 + 5b_{m+1}$. We have $b_1 = 100, a_1 = 3, a_2 = 27, a_3 = 3^{27}$, and $2 + 5b_1 = 502 < 729 = 3^6 < 3^{27}$, so $a_3 > 2 + 5b_1$. We find that $a_n > 2 + 5b_{n-2}$ for all $n \geq 3$. In particular, $a_n \geq b_{n-2}$.

On the other hand, $a_n < b_m$ implies $a_{n+1} = 3^{a_n} < 100^{b_m} = b_{m+1}$, which combined with $a_2 < b_1$ yields $a_n < b_{n-1}$ for all $n \geq 2$. Hence $b_{n-2} < a_n < b_{n-1}$ which implies that $m = n - 1$, which implies that $m = n - 1$, and for $n = 100, m = 99$. □

Problem 3. Define the sequence $(a_n)_n$ recursively by $a_1 = 2, a_2 = 5$ and

$$a_{n+1} = (2 - n^2)a_n + (2 + n^2)a_{n-1} \quad n \geq 2$$

Do there exist indices p, q, r such that $a_p \cdot a_q = a_r$?

Proof. Note that a perfect square is congruent to 0 or to 1 modulo 3. Using this fact we can easily prove by induction that $a_n \equiv 2 (mod\, 3)$, the question has a negative answer. □