## Math Circle: Rational Tangles

Don't be shy to ask questions during the exercise session, some of these questions are quite challenging. We are happy to help!

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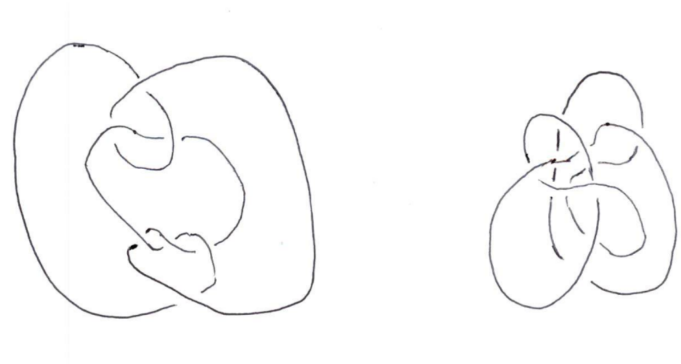
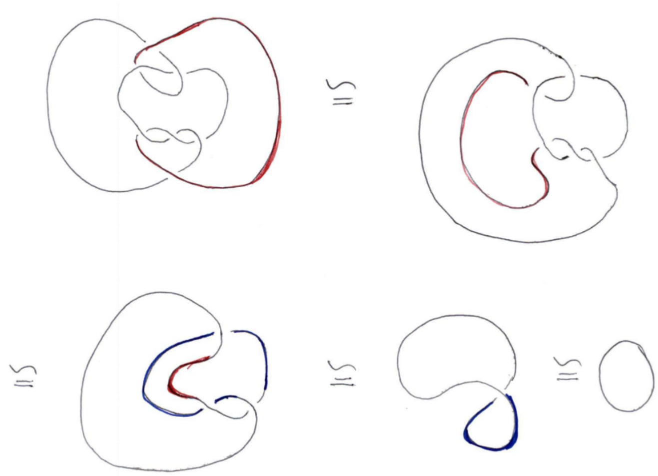
1. Try to remove as many crossings as possible from the following two knots.

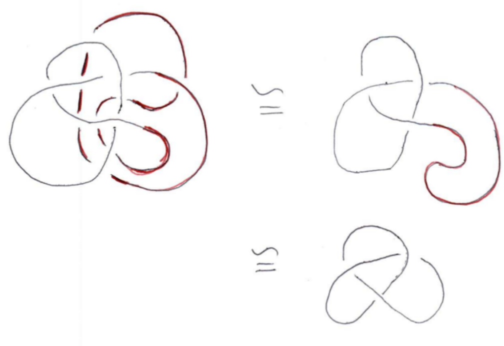


**Solution:**

For the first knot we have the following sequence of moves that show that this is actually the unknot. To improve clarity, we have colored the parts of the knot that we move.



The second knot can be simplified in the following way

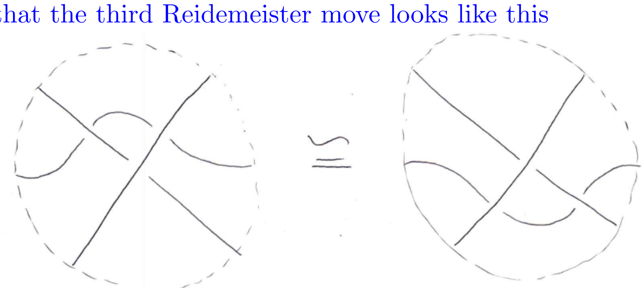


This knot is called the (right-handed) trefoil knot. We will show in *c.*) that the trefoil knot is different from the unknot. It might come as a surprise, but if we took the trefoil and replaced all overcrossings with undercrossing and vice versa, we would get a different knot (the left-handed trefoil knot). It is fun to pick a piece of string and convince yourself that this is really not the same. It is not trivial to prove this, usually one would define yet another invariant, the Jones polynomial, to distinguish the two trefoils.

2. We will now show that the two knots in the previous exercise are not the same. For this we will show that one of them is 3-colorable, while the other is not.

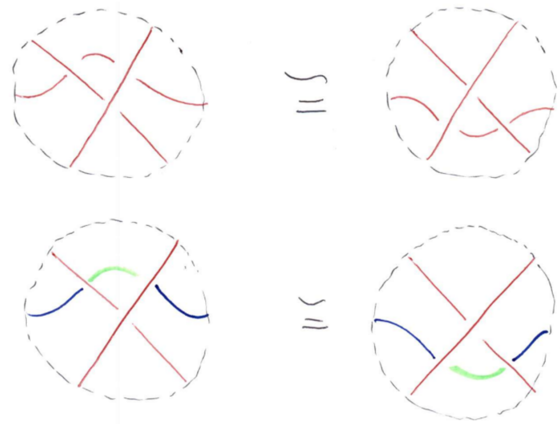
   (a) In class we have discussed that 3-colorability is preserved under the first two Reidemeister moves. Check that it is also preserved under the third Reidemeister move.

**Solution:** Recall that the third Reidemeister move looks like this

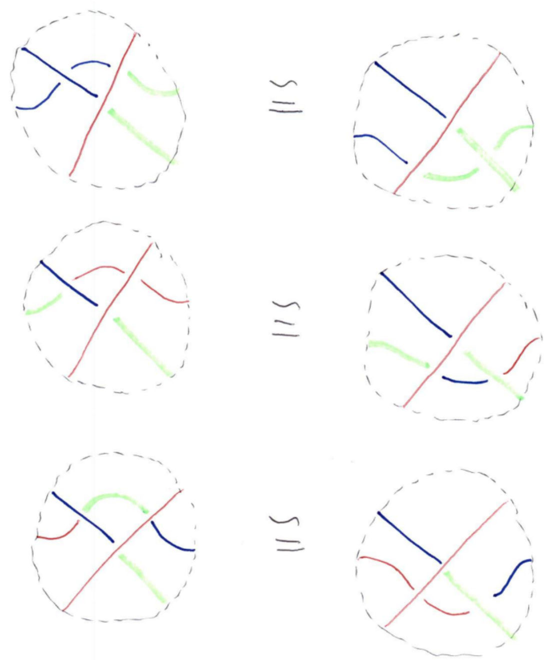


We need to show that if we can color the left picture following the rules of 3-colorability and we fix the colors of the strands that intersect the circle, then we can still color the right picture according to the rules of 3-colorability. For this we distinguish a couple of cases.

I like red, blue and green and thus this are the 3-colors I will be using (you can use whatever colors you like, just fix them beforehand). Here is a list of all possible configuration (up to permutation of colors). The first two show the case when the strands on top are both of the same color.

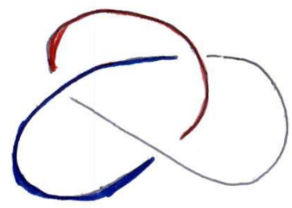


The remaining three cases deal with the case when the top strands have different colors and we just pick the three possible colors for the leftmost part of the strand on the bottom.



(b) Use your simplified knots from the first exercise to show that the first one is not 3-colorable, while the second one is. Conclude that they cannot be the same knot.

**Solution:** The unknot has no crossings, so we can use at most one color and therefore it cannot be 3-colorable. On the other hand, we can color the (right-handed) trefoil like this



As 3-colorability is an invariant, we get that the unknot and the (right-handed) trefoil are different knots.

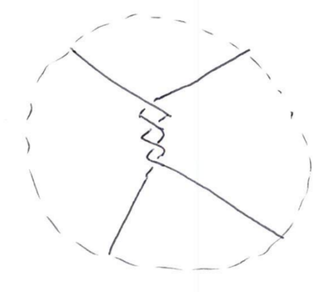
(c) Similarly as for 3-colorability, one can introduce the notion of 4-colorability. We say that a know is 4-colorable if at each crossing we have either three different colors or all the same color and we have used in total exactly 4 colors. By the same reasoning one gets that 4-colorability is a knot invariant. Show that none of the knots in the first exercise are 4-colorable, but that the following knot is 4-colorable.

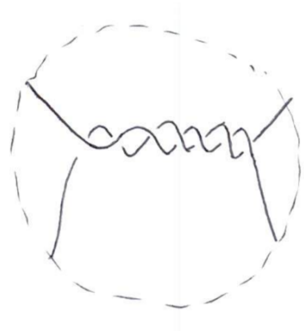
**Solution:** The trefoil knot has only 3 components that we can color and thus, it is not 4-colorable. Similarly, the unknot has only one component that we can color and hence is also not 4-colorable. On the other hand, we can color the figure 8 knot in the following manner

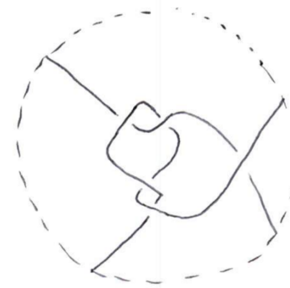
Thus, the unknot, the trefoil knot and the figure 8 knot are all different knots.

3. In this question we are going to have some fun with rational tangles. It's entertaining to play around with them and compute some of the associated continued fractions.

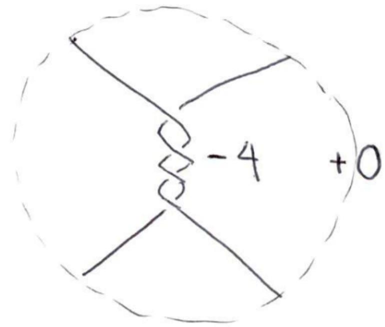
   (a) Compute the associated fractions of the following rational tangles:







**Solution:** For the first tangle, there is no vertical crossings at the end and so we have

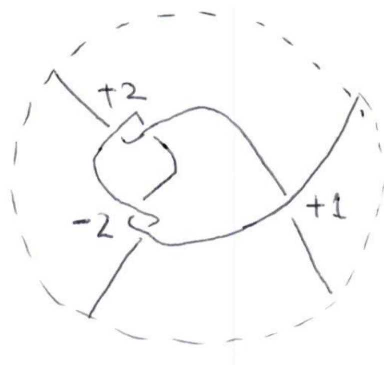



Hence, the associated fraction is

$$0 + \frac{1}{-4} = -\frac{1}{4}.$$

For the second tangle, we note that we can simplify some of the crossings by pulling the strands straight. We get


and therefore, the associated fraction is $-5$.
Finally, after pulling the strands straight, we get for the last tangle




Thus, the associated fractions is

$$1 + \frac{1}{-2 + \frac{1}{2}} = 1 + \frac{2}{-4+1} = 1 + \frac{2}{-3} = \frac{1}{3}.$$

(b) Compute the regular continued fractions associated to the rational tangles above. Use this to find a rational tangle with the same associated continued fraction and such that all crossings (except potentially the last horizontal one) are positive. Convince yourself that these really represent the same tangle that we started with.

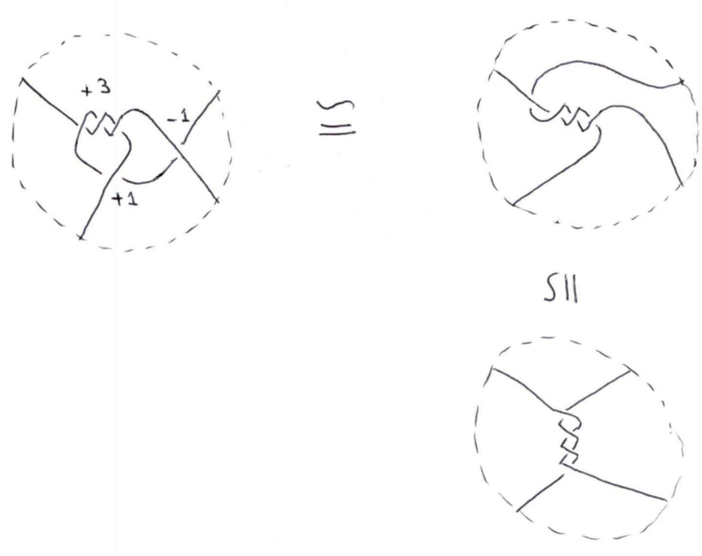
**Solution:** In the talk we saw the formula (for $b \neq 0$ and $b \neq 1$)

$$a - \frac{1}{b} = (a-1) + \frac{1}{1 + \frac{1}{b-1}}.$$

For $a = 0, b = 4$ we get

$$0 - \frac{1}{4} = -1 + \frac{1}{1 + \frac{1}{4-1}} = -1 + \frac{1}{1 + \frac{1}{3}}.$$

Hence, our tangle can represented by the following other tangle.




The second tangle is already in the desired form, we have nothing to do.

Finally, for the third one, we can either note that

$$\frac{1}{3} = 0 + \frac{1}{3}$$

or we can go through the algorithm we saw in the talk. Namely,

$$1 + \frac{1}{-2 + \frac{1}{2}} = 1 - \frac{1}{2 - \frac{1}{2}}.$$

We now use again the formula above, with $a = 2, b = 2$ and get

$$2 - \frac{1}{2} = (2 - 1) + \frac{1}{1 + \frac{1}{2-1}} = 1 + \frac{1}{2}.$$

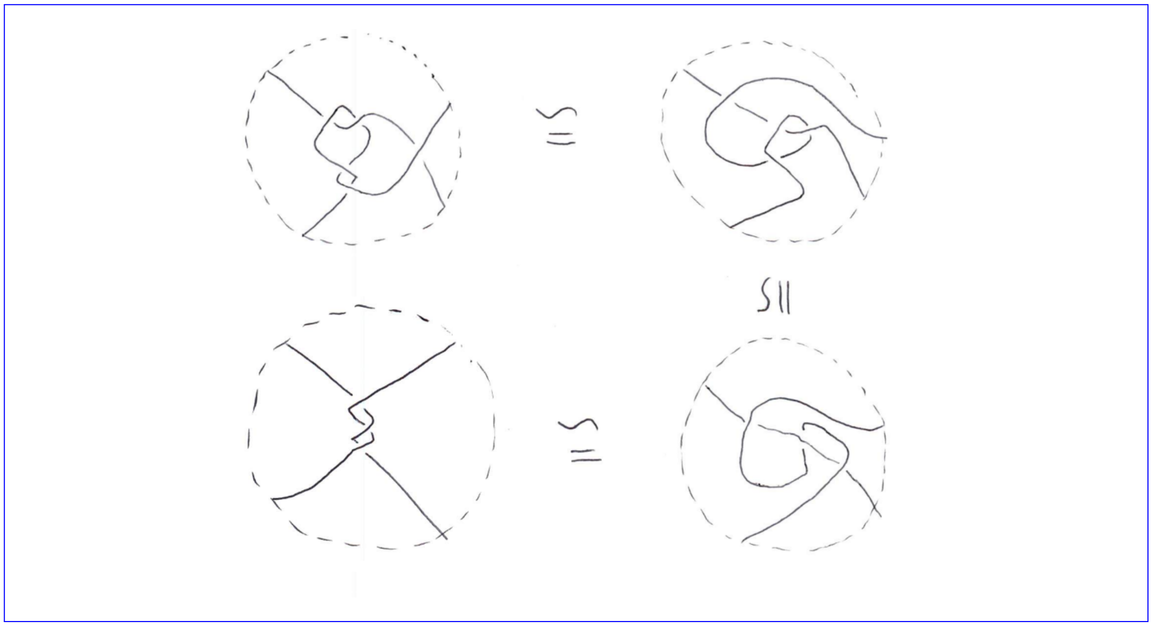
Hence, we get

$$1 - \frac{1}{2 - \frac{1}{2}} = 1 - \frac{1}{(1 + \frac{1}{2})}.$$

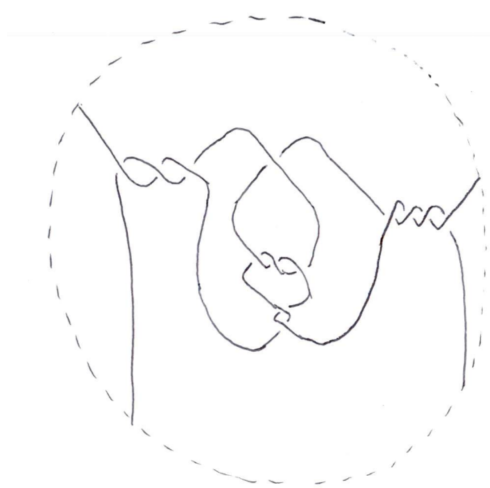
Now we use again the formula, with $a = 1, b = 1 + \dfrac{1}{2}$ and get

$$1 - \frac{1}{(1 + \frac{1}{2})} = (1 - 1) + \frac{1}{1 + \frac{1}{1 - 1 + \frac{1}{2}}} = 0 + \frac{1}{1 + 2} = 0 + \frac{1}{3}.$$

Hence, this gives us the regular representation

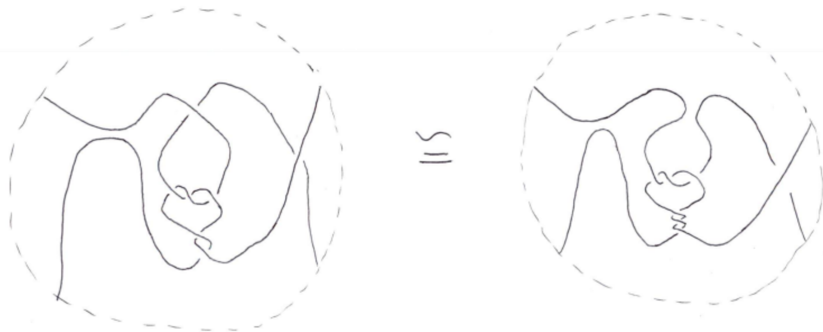


(c) Transform the tangle below into a rational tangle and compute its associated continued fraction.

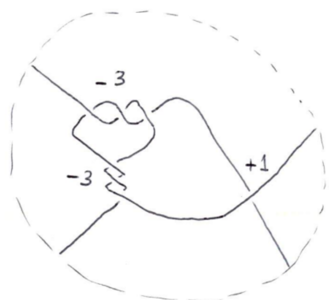


Hint: It might help to fix four points and then rotate a big chunk of the tangle to remove undesirable crossings.

**Solution:** We rotate three times around the horizontal axis to remove the first three crossings in the beginning. Afterwards we rotate once along the vertical axis to remove another undesirable crossing. This yields



Pulling the strangs straight gives us




Hence, the associated continued fraction is

$$1 + \frac{1}{-3 + \frac{1}{-3}} = 1 - \frac{1}{3 + \frac{1}{3}} = 1 - \frac{3}{10} = \frac{7}{10}.$$

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Knot theory plays a role in certain branches of chemistry, especially tertiary structures of complicated bio-molecules. Some people turned these problems into a game. If you want to have some more fun with knot theory and contribute to protein folding, you can check out https://fold.it/. We hope that you enjoyed our brief journey into the world of mathematics and see you again next time.