UBC Math Circle 2023 Problem Set 8

1. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$ using this operation?

   Proof. Because
$$a^2 + b^2 = (\frac{a+b}{\sqrt{2}})^2 + (\frac{a-b}{\sqrt{2}})^2$$
   the sum of the squares of the numbers in a triple is invariant under the operation. The sum of squares of the first triple is $\frac{9}{2}$ and that of the second is $6 + 2\sqrt{2}$, so the first triple cannot be transformed into the second. □

2. Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

   Proof. There are only finitely many polygonal lines with these points as vertices. Choose the one of minimal length $P_1P_2 \cdots P_n$. If two sides, say P_iP_{i+1} and P_jP_{j+1}, intersect at some point M, replace them by P_iP_j and $P_{i+1}P_{j+1}$ to obtain the closed polygonal line $P_1 \cdots P_iP_jP_{j-1} \cdots P_{i+1}P_{j+1} \cdots P_n$, by the following figure:

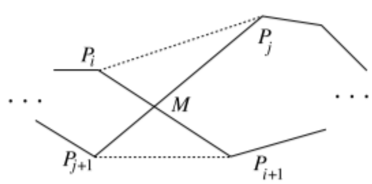
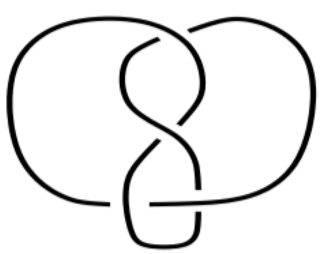


Figure 52

   The triangle inequality in triangles MP_iP_j and $MP_{i+1}P_{j+1}$ shows that this polygonal line has shorter length, a contradiction. It follows that $P_1P_2 \cdots P_n$ has no self-intersections, as desired. □

3. Prove that the figure eight knot in the Figure is knotted.



   Proof. The invariant is the 5-colorability of the knot, i.e., the property of a knot to admit a coloring by the residue classes modulo 5 such that

- At least two residue classes are used;
- At each crossing, $a + c \equiv 2b(\mod 5)$, where b is the residue class assigned to the overcrossing, and a and c are the residue classes assigned to other two arcs.

A coloring of the figure eight knot is given below, while the trivial knot does not admit 5-colorings since its simplest diagram does not. This proves that the figure eight knot is knotted.

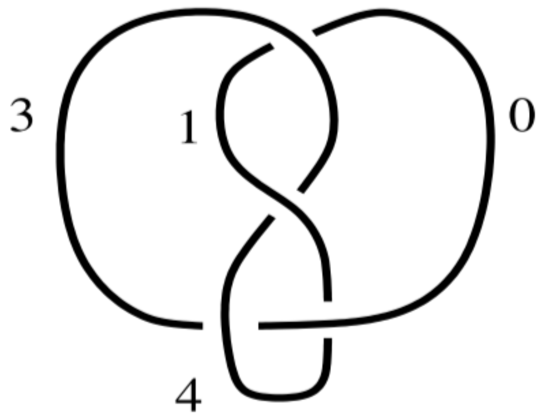


Figure 56

□