UBC MATH CIRCLE 2024 PROBLEM SET 10 SOLUTIONS

Problem 1. *Determine all roots of the system of equations*

$$x + y = z = 3$$
$$x^2 + y^2 + z^2 = 3$$
$$x^3 + y^3 + z^3 = 3$$

Solution. Note that

$$9 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz) = 3 + 2(xy + xz + yz)$$

Thus $xy + xz + yz = 3$. Similarly,

$$3 - 3xyz = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz) = 0$$

Thus $xyz = 1$. Now consider the polynomial $P(t) = (t-x)(t-y)(t-z)$. By Vieta's formulas,

$$P(t) = t^3 - 3t^2 + 3t - 1 = (t - 1)^3$$

Since the roots of $P(t)$ are $\{x, y, z\}$, we conclude that $x = y = z = 1$.

Problem 2. *Find all polynomials $P(x)$ with real coefficients such that $P(x)P(x+1) = P(x^2)$ for all $x \in \mathbb{R}$.*

Proof. Suppose first that $P(x) \equiv a$ is constant. Then $a^2 = a$ and $a = 0, 1$. This gives two solutions $P(x) \equiv 0$ and $P(x) \equiv 1$. Hence assume that $P(x)$ is not constant. If z is a root of $P(x)$, then $(z^2) = P(z)P(z+1) = 0$. So z^2 is a root of $P(x)$. In particular, if $P(x)$ has a root of absolute value $|z| \neq 0, 1$, then the sequence $\{z, z^2, z^4, \ldots\}$ contains complex numbers of distinct absolute values. Thus $P(x)$ has infinitely many roots and $P(x) \equiv 0$, contradiction. Hence any root $z \neq 0$ of $P(x)$ must have absolute value 1. If $z \neq 0$ is a root of $P(x)$, then $P((z - 1)^2) = P(z - 1)P(z) = 0$, so $(z - 1)^2$ is a root of $P(x)$. Thus $|z - 1| = 0, 1$. So either $z = 1$ or z lies on the circles $|z - 1| = 1$ and $|z| = 1$, so $z = e^{\pm i\pi/3}$. In the latter two cases, we conclude that $z_1 = (z - 1)^2 = e^{\pm 2\pi i/3}$ is also a root of $P(x)$. By the same argument, $(z_1 - 1)^2$ is a root, but this is impossible since $|z_1 - 1| > 1$. Thus the only roots of $P(x)$ are $0, 1$ and $P(x) = cx^n(x - 1)^m$. Plugging this back into the original equation, we find that

$$c^2x^n(x - 1)^m(x + 1)^n x^m = x^{n+m}(x - 1)^m(x + 1)^n = cx^{2n}(x^2 - 1)^m$$

Thus we conclude that $n = m$ and $c = 0, 1$. Hence the only solutions are:

$$P(x) = \begin{cases} 0 & x \in \mathbb{R} \\ 1 & x \in \mathbb{R} \\ x^n(x - 1)^n & x \in \mathbb{R}, n \in \mathbb{N} \end{cases}$$

□

Problem 3. *A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one of the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.*

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Solution. Fix the corners of $\mathcal{R}$ to be $(0,0),(m,0),(0,n)$ and (m,n). Evidently all vertices of the smaller rectangles will lie at integer points. Denote by $[x,y]$ the square with vertices $\{(x,y),(x+1,y),(x,y+1),(x+1,y+1)\}$. Colour the square $[x,y]$ black if $x+y$ is even and white otherwise. Note that the squares in the corner of $\mathcal{R}$ are black since both m,n are odd. Evidently, a small rectangle $\mathcal{R}'$ in $\mathcal{R}$ satisfies the condition of the problem if all four squares in the corners are black, or equivalently, $\mathcal{R}'$ contains more black rectangles than white rectangles. Since the small rectangles partition the squares contained in $\mathcal{R}$ and since $\mathcal{R}$ contains more black squares than white squares, it follows that some small rectangle in $\mathcal{R}$ must similarly satisfy this condition.